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Non-local reductions of multi-component NLS equations

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PLAN

- The inverse scattering method and NLS-type eqs.
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- Effects of reductions on the scattering matrix
- RHP and multi-component GI equations
- Soliton solutions
- Integrals of motion of the multi-component NLS equations
- Conclusions and open questions

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Introduction

The Zakharov-Shabat systems and the NLS eq.:

$$L\psi(x,t,\lambda) \equiv i\frac{\partial \psi}{\partial x} + (Q(x,t) - \lambda\sigma_3)\psi(x,t,\lambda) = 0,$$

$$Q(x,t) = \begin{pmatrix} 0 & q \\ \epsilon q^* & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathrm{i}q_t + q_{xx} + \epsilon|q|^2 q(x,t) = 0.$$

Quadratic bundle and the Kaup-Newell equation (1978):

$$\mathrm{i}q_t + q_{xx} + \epsilon\mathrm{i}(|q|^2 q)_x = 0.$$

$$L(\lambda) \quad = \quad \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2\sigma_3; \quad M(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{3} V_k(x,t)\lambda^{4-k} - \lambda^4\sigma_3$$

$$
\begin{aligned}
V_1(x,t) &= Q(x,t), \quad V_2(x,t) = \frac{1}{2}\epsilon|q^2(x,t)|\sigma_3, \\
V_3(x,t) &= \frac{\mathrm{i}}{2}\sigma_3 Q_x(x,t) + \epsilon|q^2(x,t)|Q(x,t), \quad V_4 = 0.
\end{aligned}
$$

Kaup-Newell eq. (1978) is related via gauge transformations to three other integrable NLEEs: the one studied by Chen-Lee-Liu (1982)

$$
\mathrm{i}q_t + q_{xx} + \mathrm{i}|q|^2 q_x = 0,
$$

V.G.-Ivanov (GI) eq. (1981)

$$
\mathrm{i}q_t + q_{xx} + \epsilon \mathrm{i} q^2 q_x^* + \frac{1}{2}|q|^4 q(x,t) = 0,
$$

is treated by the Lax operator

$$
L(\lambda) = \mathrm{i}\partial_x + \frac{\epsilon}{2}|q|^2\sigma_3 + \lambda Q(x,t) - \lambda^2\sigma_3,
$$

quadratic in λ and related to the algebra $sl(2,\mathbb{C})$:

NLEE related to graded Lie algebras and symmetric spaces

$\mathbb{Z}_2$-graded Lie algebras, Helgasson:

$$\mathfrak{g} \simeq \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, \quad \mathfrak{g}^{(0)} \equiv \{X \in \mathfrak{g},\ [J,X]=0\}, \quad \mathfrak{g}^{(1)} \equiv \{Y \in \mathfrak{g},\ JY + YJ = 0\},$$

where $J = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}$. $\mathfrak{g}^{(1)}$ is the co-adjoint orbit passing through J — phase space for our NLEE.

Now the Lax operators that are given by:

$$L\psi \equiv i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda Q(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \qquad Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix},$$

$$M\psi \equiv i\frac{\partial \psi}{\partial t} + (V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J)\psi(x,t,\lambda) = 0.$$

where $Q(x,t)$, $V_3(x,t) \in \mathfrak{g}^{(1)}$ and $U_2(x,t)$, $V_2(x,t)$ and $V_4(x,t) \in \mathfrak{g}^{(0)}$. They are given by:

$$V_1 = Q(x,t), \qquad V_2 = \frac{1}{2}\begin{pmatrix} \boldsymbol{qp} & 0 \\ 0 & -\boldsymbol{pq} \end{pmatrix}, \qquad V_3 = \frac{i}{2}\begin{pmatrix} 0 & \boldsymbol{q}_x \\ -\boldsymbol{p}_x & 0 \end{pmatrix},$$

$$V_4 = \frac{1}{4} \left(\begin{array}{cc} i(\boldsymbol{q}_x \boldsymbol{p} - \boldsymbol{q}\boldsymbol{p}_x) + \frac{1}{2}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p} & 0 \\ 0 & -i(\boldsymbol{p}\boldsymbol{q}_x - \boldsymbol{p}_x \boldsymbol{q}) - \frac{1}{2}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q} \end{array} \right).$$

we get the multicomponent GI eq.:

$$i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i}{2}\boldsymbol{q}\frac{\partial \boldsymbol{p}}{\partial x}\boldsymbol{q} + \frac{1}{4}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q} = 0,$$
$$-i\frac{\partial \boldsymbol{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{p}}{\partial x^2} + \frac{i}{2}\boldsymbol{p}\frac{\partial \boldsymbol{q}}{\partial x}\boldsymbol{p} + \frac{1}{4}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p} = 0.$$

Simple gauge transformation leads to the Lax pair:

$$\tilde{L}\tilde{\psi} \equiv i\frac{\partial \tilde{\psi}}{\partial x} + (\lambda \tilde{Q}(x,t) - \lambda^2 J)\tilde{\psi}(x,t,\lambda) = 0, \qquad \tilde{Q}(x,t) = \left(\begin{array}{cc} 0 & \tilde{\boldsymbol{q}} \\ \tilde{\boldsymbol{p}} & 0 \end{array} \right),$$
$$\tilde{M}\tilde{\psi} \equiv i\frac{\partial \tilde{\psi}}{\partial t} + (\lambda \tilde{V}_3(x,t) + \lambda^2 \tilde{V}_2(x,t) + \lambda^3 \tilde{Q}(x,t) - \lambda^4 J)\tilde{\psi}(x,t,\lambda) = 0.$$

which allows one to solve the multicomponent Kaup-Newell system:

$$
\begin{aligned}
i\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{q}}}{\partial x^2} + i\frac{\partial \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}}{\partial x} = 0, \\
-i\frac{\partial \tilde{\boldsymbol{p}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{p}}}{\partial x^2} - i\frac{\partial \tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}}{\partial x} = 0.
\end{aligned}
$$

Examples of local reductions

All these equations allow the reduction:

$$
\boldsymbol{p} = \pm \boldsymbol{q}^\dagger \qquad \text{and} \qquad \tilde{\boldsymbol{p}} = \pm \tilde{\boldsymbol{q}}^\dagger .
$$

The inverse scattering method and NLS-type eqs.

The inverse scattering method for the NLS eq. – Zakharov, Shabat (1971); for the N-wave equations – Zakharov, Shabat, Manakov (1973–1974)..

Lax representation:

$$[L, M] \equiv 0,$$
$$L\psi \equiv i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda Q(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,$$
$$M\psi \equiv i\frac{\partial \psi}{\partial t} + (V(x,t,\lambda) - \lambda^4 J)\psi(x,t,\lambda) = 0,$$

where J is a constant block-diagonal matrix.

λ^6 a) $[J, J] = 0,$ λ^5 b $[J, Q] + [Q, J] = 0,$

$\lambda^4, \cdots, \lambda^2$ Identities ,

$$\lambda \quad \text{f)} \quad iQ_{1,t} - iV_{3,x} + [V_4, Q] + [V_3, U_2] = 0,$$
$$\lambda^0 \quad \text{g)} \quad iU_{2,t} - iV_{4,x} + [V_4, U_2] = 0.$$

Eq. f) provides, say GI eq; Eq. g) is satisfied as a consequence of GI eq. Therefore, if $\boldsymbol{q}(x,t)$ and $\boldsymbol{p}(x,t)$ satisfy GI eq., then $[L, M] = 0$ **identically with respect to** λ.

Solving Nonlinear Cauchy problems by the ISM

Find solution to the GI eqs. such that for $t = 0$

$$Q(x, t = 0) = Q_0(x).$$

$$\begin{array}{ccccccc} Q_0 & \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow & Q(x,t) \\ & & \mathrm{I}\downarrow & & \uparrow\mathrm{III} & & \\ & & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & & \end{array} \tag{1}$$

Step I: Given $Q_1(x, t = 0) = Q_0(x)$ construct the scattering matrix $T(\lambda, 0)$. Jost solutions:

$$L\phi(x, \lambda) = 0, \qquad \lim_{x\to-\infty} \phi(x, \lambda)e^{i\lambda^2 Jx} = \mathbb{1},$$

$$L\psi(x, \lambda) = 0, \qquad \lim_{x\to\infty} \psi(x, \lambda)e^{i\lambda^2 Jx} = \mathbb{1},$$

$$T(\lambda, 0) = \psi^{-1}(x, \lambda)\phi(x, \lambda).$$

Step II: From the Lax representation there follows:

$$i\frac{\partial T}{\partial t} - \lambda^4[J, T(\lambda, t)] = 0,$$

i.e.

$$T(\lambda, t) = e^{-i\lambda^4 Jt} T(\lambda, 0) e^{i\lambda^4 Jt}.$$

Indeed: From $[L, M]\phi = 0$ it follows

$$LM\phi(x,t,\lambda) - ML\phi(x,t,\lambda) = LM\phi(x,t,\lambda), \qquad \text{or} \qquad M\phi(x,t,\lambda) = \phi(x,t,\lambda)C(\lambda).$$

$$\lim_{x\to -\infty} M\phi(x,t,\lambda) = \lim_{x\to -\infty} \left(i\frac{\partial}{\partial t} - \lambda^4 J \right) e^{-i\lambda^4 Jx} = e^{-i\lambda^4 Jx} C(\lambda),$$

i.e.

$$C(\lambda) = -\lambda^4 J.$$

Next

$$\lim_{x\to \infty} M\phi(x,t,\lambda) = \lim_{x\to -\infty} \left(i\frac{\partial}{\partial t} - \lambda^4 J \right) T(\lambda, t) e^{-i\lambda^4 Jx} = T(\lambda, t) e^{-i\lambda^4 Jx} C(\lambda),$$

Put $T(\lambda,t) = \begin{pmatrix} \boldsymbol{a}^+ & -\boldsymbol{b}^- \\ \boldsymbol{b}^+ & \boldsymbol{a}^- \end{pmatrix}$. Then

$$i\frac{\partial T}{\partial t} - \lambda^4[J, T(\lambda,t)] = 0 \quad \Leftrightarrow \quad \begin{array}{l} i\frac{\partial \boldsymbol{a}^\pm}{\partial t} = 0, \\ i\frac{\partial \boldsymbol{b}^\pm}{\partial t} \pm 2\lambda^4 b^\pm(\lambda,t) = 0 \end{array}.$$

Two important consequences:
GI eq. has an infinite number of conserved quantities;
GI eq. can be linearized globally.

Step III: Given $T(\lambda,t)$ construct the potential $Q_1(x,t)$ for $t>0$.
For $\mathfrak{g} \simeq sl(2)$ – GLM eq. – Volterra type integral equations
It can be generalized also for 2×2 block-matrix valued Lax operators.

Now we are using more effective method for solving the ISP by reducing it to Riemann-Hilbert problem.

Important: All steps reduce to **linear** integral equations. Thus the nonlinear Cauchy problem reduces to a sequence of three **linear Cauchy problems**; each has unique solution!

The direct scattering problem for L

C1: $Q(x,t)$ are smooth enough and fall off to zero fast enough for $x \to \pm\infty$ for all t.

C2: $Q(x,t)$ is such that L has at most finite number of simple discrete eigenvalues.

The Jost solutions of L are defined by their asymptotics at $x \to \pm\infty$:

$$\lim_{x\to\infty} \psi(x,\lambda)e^{i\lambda^2 Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda)e^{i\lambda^2 Jx} = \mathbb{1},$$

Along with the Jost solutions, we introduce

$$X_+(x,\lambda) = \psi(x,\lambda)e^{i\lambda^2 Jx}, \qquad X_-(x,\lambda) = \phi(x,\lambda)e^{i\lambda^2 Jx};$$

which satisfy the following linear integral equations

$$X_\pm(x,\lambda) \quad = \mathbb{1} + i\int_{\pm\infty}^{x} dy e^{-i\lambda^2 J(x-y)} Q(y) X_\pm(y,\lambda) e^{i\lambda^2 J(x-y)}.$$

$$\psi(x,\lambda) = \left(|\psi^-(x,\lambda)\rangle, |\psi^+(x,\lambda)\rangle\right), \qquad \phi(x,\lambda) = \left(|\phi^+(x,\lambda)\rangle, |\phi^-(x,\lambda)\rangle\right),$$

where + means analyticity for $\lambda \in \Omega_1 \cup \Omega_3$
and − means analyticity for $\lambda \in \Omega_2 \cup \Omega_4$

The scattering matrix $T(\lambda)$ and its inverse $\hat{T}(\lambda)$:

$$\phi(x,\lambda) = \psi(x,\lambda)T(\lambda), \qquad T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & -\boldsymbol{b}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & \boldsymbol{a}^-(\lambda) \end{pmatrix}$$

$$\psi(x,\lambda) = \phi(x,\lambda)\hat{T}(\lambda), \qquad \hat{T}(\lambda) \equiv \begin{pmatrix} \boldsymbol{c}^-(\lambda) & \boldsymbol{d}^-(\lambda) \\ -\boldsymbol{d}^+(\lambda) & \boldsymbol{c}^+(\lambda) \end{pmatrix},$$

$$\boldsymbol{a}^+(\lambda), \quad \boldsymbol{c}^+(\lambda) \quad \text{are analytic functions of } \lambda \text{ for} \quad \lambda \in \Omega_1 \cup \Omega_3,$$

$$\boldsymbol{a}^-(\lambda), \quad \boldsymbol{c}^-(\lambda) \quad \text{are analytic functions of } \lambda \text{ for} \quad \lambda \in \Omega_2 \cup \Omega_4.$$

Reflection coefficients $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ (not analytic):

$$\rho^\pm(\lambda) = \boldsymbol{b}^\pm \hat{\boldsymbol{a}}^\pm(\lambda) = \hat{\boldsymbol{c}}^\pm \boldsymbol{d}^\pm(\lambda), \qquad \tau^\pm(\lambda) = \hat{\boldsymbol{a}}^\pm \boldsymbol{b}^\mp(\lambda) = \boldsymbol{d}^\mp \hat{\boldsymbol{c}}^\pm(\lambda),$$

We will need also the asymptotics for $\lambda \to \infty$:

$$\lim_{\lambda\to-\infty} \phi(x,\lambda)e^{i\lambda Jx} = \lim_{\lambda\to\infty} \psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{\lambda\to\infty} T(\lambda) = \mathbb{1},$$

i.e. $\lim_{\lambda\to\infty} \boldsymbol{a}^{\pm}(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^{\pm}(\lambda) = \mathbb{1}$.

The inverse to the Jost solutions $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ are solutions to:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,\lambda)(U_2(x,t) + \lambda Q(x,t) - \lambda^2 J) = 0,$$

satisfying the conditions:

$$\lim_{x\to\infty} e^{-i\lambda J x}\hat{\psi}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} e^{-i\lambda J x}\hat{\phi}(x,\lambda) = \mathbb{1}.$$

As a result the sets of rows of $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ are analytic in λ:

$$\hat{\psi}(x,\lambda) = \begin{pmatrix} \langle\hat{\psi}^{+}(x,\lambda)| \\ \langle\hat{\psi}^{-}(x,\lambda)| \end{pmatrix}, \qquad \hat{\phi}(x,\lambda) = \begin{pmatrix} \langle\hat{\phi}^{-}(x,\lambda)| \\ \langle\hat{\phi}^{+}(x,\lambda)| \end{pmatrix},$$

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Figure 1: The continuous

Reductions of polynomial bundles

Local Reductions

An important and systematic tool to construct new integrable NLEE is the Mikhailov reduction group (1981):

$$
\begin{array}{lll}
1) & A_1 U^\dagger(x,t,\kappa_1\lambda^*) A_1^{-1} = U(x,t,\lambda), & A_1 V^\dagger(x,t,\kappa_1\lambda^*) A_1^{-1} = V(x,t,\lambda), \\
2) & A_2 U^T(x,t,\kappa_2\lambda) A_2^{-1} = -U(x,t,\lambda), & A_2 V^T(x,t,\kappa_2\lambda) A_2^{-1} = -V(x,t,\lambda), \\
3) & A_3 U^*(x,t,\kappa_1\lambda^*) A_3^{-1} = -U(x,t,\lambda), & A_3 V^*(x,t,\kappa_1\lambda^*) A_3^{-1} = -V(x,t,\lambda), \\
4) & A_4 U(x,t,\kappa_2\lambda) A_4^{-1} = U(x,t,\lambda), & A_4 V(x,t,\kappa_2\lambda) A_4^{-1} = V(x,t,\lambda).
\end{array}
$$

The consequences of the reductions 1) and 3) on the NLEE are:

$$
\begin{array}{llll}
1) & A_1 J A_1^{-1} = J, & \kappa_1 A_1 Q^\dagger A_1^{-1} = Q(x,t), & A_1 U_2^\dagger(x,t) A_1^{-1} = U_2(x,t), \\
3) & A_3 J A_3^{-1} = -J, & \kappa_3 A_3 Q^* A_3^{-1} = -Q(x,t), & A_3 U_2^*(x,t) A_3^{-1} = -U_2(x,t),
\end{array}
$$

where $\kappa_1^2 = \kappa_3^2 = 1$ and $A_1^2 = A_3^2 = \mathbb{1}$. From $A_1 J A_1^{-1} = J$ (resp. $A_3 J A_3^{-1} = -J$) we find that A_1 is block-diagonal (resp. A_3 is block-off-diagonal) matrix. If we introduce

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix},$$

we obtain

$$\begin{aligned} 1) \qquad & \kappa_1 a_1 \boldsymbol{p}^\dagger \hat{a}_2 = \boldsymbol{q}, \qquad && \kappa_1 a_2 \boldsymbol{q}^\dagger \hat{a}_1 = \boldsymbol{p}, \\ & A_1 U_2^\dagger A_1^{-1} = U_2 \\ 3) \qquad & \kappa_3 b_1 \boldsymbol{p}^* \hat{b}_2 = -\boldsymbol{q}, \qquad && \kappa_3 b_2 \boldsymbol{q}^* \hat{b}_1 = -\boldsymbol{p}, \\ & A_3 U_2^* A_3^{-1} = -U_2. \end{aligned}$$

As a result we get a multicomponent GI equation:

$$i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i\kappa_1}{2}\boldsymbol{q} a_2 \frac{\partial \boldsymbol{q}^\dagger}{\partial x}\hat{a}_1 \boldsymbol{q} + \frac{1}{4}\boldsymbol{q} a_2 \boldsymbol{q}^\dagger \hat{a}_1 \boldsymbol{q} a_2 \boldsymbol{q}^\dagger \hat{a}_1 \boldsymbol{q} = 0.$$

while the equation Kau–Newell eq. goes into a multicompomemt KN equation:

$$i\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} + \frac{\partial^2 \tilde{\boldsymbol{q}}}{\partial x^2} + i\kappa_1 \frac{\partial}{\partial x}\left(\tilde{\boldsymbol{q}} a_2 \tilde{\boldsymbol{q}}^\dagger \hat{a}_1 \tilde{\boldsymbol{q}}\right) = 0.$$

Non-Local Reductions

The idea starts from quantum mechanics where special classes of potentials like . the $\mathcal{PT}$-symmetric ones

$$V(x,t) = \psi(x,t)\psi^*(-x,-t).$$

became important. These systems find applications in Nonlinear Optics.

Supposing that the wave function is a scalar, this leads to the following action of the operator of spatial reflection on the space of states:

$$\mathcal{P}\psi(x,t) = \psi(-x,t).$$

Similar arguments apply also to the time reversal operator $\mathcal{T}$:

$$\mathcal{T}\psi(x,t) = \psi^*(x,-t).$$

Therefore, the Hamiltonian and the wave function are $\mathcal{PT}$-symmetric, if

$$\mathcal{H}(x,t) = \mathcal{H}^*(-x,-t), \qquad \psi(x,t) = \psi^*(-x,-t).$$

In addition – charge conjugation symmetry (particle-antiparticle symmetry) $\mathcal{C}$:

$$\mathcal{C}\mathcal{H}^*(x,t) = \mathcal{H}(x,t), \qquad \mathcal{C}\psi^*(x,t) = \psi(x,t).$$

The $\mathcal{C}$-symmetry can be realized by an unitary linear operator, see Peskin (1995). The Hamiltonian and the wave function are $\mathcal{C}\mathcal{P}\mathcal{T}$-symmetric, if

$$\mathcal{H}(x,t) = \mathcal{H}(-x,-t), \qquad \psi(x,t) = \psi(-x,-t).$$

Integrable systems with $\mathcal{P}\mathcal{T}$-symmetry were studied extensively over the last two decades Fring (2007).

The Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + V(x,t)\psi(x,t) = E\psi(x,t).$$

There are situations when $V(x,t) \simeq \psi(x,t)\psi^*(x,t)$. Then

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + |\psi(x,t)|^2\psi(x,t) = E\psi(x,t).$$

Put $u(x,t) = e^{-iEt}\psi(x,t)$ and NLS eq. with local reduction follows:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + |u(x,t)|^2 u(x,t) = 0.$$

Obviously also the non-local reductions can be applied.

It is important to note, that for the derivative NLS equations *there are no* reductions compatible with either $\mathcal{P}$- or $\mathcal{T}$-symmetry separately. However the $\mathbb{Z}_2$ reductions

$$\begin{aligned}
&1) \quad C_1 U^\dagger(-x,-t,\kappa_1\lambda^*)C_1^{-1} = -U(x,t,\lambda), \quad C_1 V^\dagger(-x,-t,\kappa_1\lambda^*)C_1^{-1} = -V(x,t,\lambda)\\
&2) \quad C_2 U^T(-x,-t,\kappa_2\lambda)C_2^{-1} = U(x,t,\lambda), \quad C_2 V^T(-x,-t,\kappa_2\lambda)C_2^{-1} = V(x,t,\lambda),\\
&3) \quad C_3 U^*(-x,-t,\kappa_1\lambda^*)C_3^{-1} = U(x,t,\lambda), \quad C_3 V^*(-x,-t,\kappa_1\lambda^*)C_3^{-1} = V(x,t,\lambda),\\
&4) \quad C_4 U(-x,-t,\kappa_2\lambda)C_4^{-1} = -U(x,t,\lambda), \quad C_4 V(-x,-t,\kappa_2\lambda)C_4^{-1} = -V(x,t,\lambda)
\end{aligned}$$

are obviously $\mathcal{PT}$-symmetric Valchev, (2008). Here $\kappa_i^2 = 1$ and A_i and C_i, $i = 1,\dots,4$ are involutive automorphisms of the relevant Lie algebra.

Now the consequences of the reductions 1) and 3) on the NLEE. It is easy to see that they restrict $U_0(x,t)$ and $Q(x,t)$ by:

$$\begin{aligned}
1)\qquad & C_1 J C_1^{-1} = -J, \qquad & \kappa_1 C_1 Q^\dagger(-x,-t) C_1^{-1} &= -Q(x,t), \\
& & C_1 U_2^\dagger(-x,-t) C_1^{-1} &= -U_2(x,t), \\
3)\qquad & C_3 J C_3^{-1} = J, \qquad & \kappa_3 C_3 Q^*(-x,-t) C_3^{-1} &= Q(x,t), \\
& & C_3 U_2^*(-x,-t) C_3^{-1} &= U_2(x,t),
\end{aligned}$$

where $\kappa_1^2 = \kappa_3^2 = 1$ and $C_1^2 = C_3^2 = \mathbb{1}$. From $C_1 J C_1^{-1} = -J$ (resp. $C_3 J C_3^{-1} = J$) we find that C_3 is block-diagonal (resp. C_1 is block-off-diagonal) matrix. If we introduce

$$C_1 = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

we obtain

1) $\kappa_1 c_1 \boldsymbol{q}^\dagger(-x,-t)\hat{c}_2 = -\boldsymbol{q}(x,t), \qquad \kappa_1 c_2 \boldsymbol{p}^\dagger(-x,-t)\hat{c}_1 = -\boldsymbol{p}(x,t),$

$C_1 U_2^\dagger(-x,-t) C_1^{-1} = -U_2(x,t)$

3) $\kappa_3 d_1 \boldsymbol{q}^*(-x,-t)\hat{d}_2 = \boldsymbol{q}(x,t), \qquad \kappa_3 d_2 \boldsymbol{p}^*(-x,-t)\hat{d}_1 = \boldsymbol{p}(x,t),$

$C_3 U_2^*(-x,-t) C_3^{-1} = U_2(x,t).$

On the Jost solutions we have

$$\phi^\dagger(x,t,\lambda^*) = \psi^{-1}(-x,t,-\lambda), \qquad \psi^\dagger(x,t,\lambda^*) = \phi^{-1}(x,t,-\lambda),$$

so for the scattering matrix we have

$$T^\dagger(t,-\lambda^*) = T(t,\lambda),$$

As a consequence for the Gauss factors we get:

$$T^{-\dagger}(-\lambda^*) = \hat{S}^+(\lambda), \qquad T^{+\dagger}(-\lambda^*) = \hat{S}^-(\lambda), \qquad D^{\pm\dagger}(\lambda^*) \;= \hat{D}^\pm(-\lambda).$$

In analogy with the local reductions, the kernel of the resolvent has poles at the points $\lambda_2^\pm$ at which $D^\pm(\lambda)$ have poles or zeroes. In particular, if

λ_2^+ is an eigenvalue, then $-\lambda_2^+$ is also an eigenvalue. For the reflection coefficients we obtain the constraints:

$$\tau^+(-\lambda) = -\rho^{+,*}(\lambda), \qquad \tau^-(-\lambda) = -\rho^{-,*}(\lambda),$$

Remark 1. *In what follows for the sake of simplicity we specify* $A_1 = C_3 = J$ *and* $A_3 = C_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. *In the latter case we restrict ourselves to the special case when* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *are square matrices, i.e. our symmetric space is* $SU(2q)/S(U(q) \otimes U(q))$.

The fundamental analytic solutions and the RHP

The next step is to construct the fundamental analytic solutions of L. In our case this is done simply by combining the blocks of Jost solutions with the same analytic properties:

$$\chi^+(x,\lambda) \equiv \left(|\phi^+\rangle, |\psi^+\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^+(\lambda) = \psi(x,\lambda)\boldsymbol{T}^-(\lambda),$$
$$\chi^-(x,\lambda) \equiv \left(|\psi^-\rangle, |\phi^-\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^-(\lambda) = \psi(x,\lambda)\boldsymbol{T}^+(\lambda),$$

where the block-triangular functions $\boldsymbol{S}^\pm(\lambda)$ and $\boldsymbol{T}^\pm(\lambda)$ are given by:

$$\boldsymbol{S}^+(\lambda) = \begin{pmatrix} \mathbb{1} & \boldsymbol{d}^-(\lambda) \\ 0 & \boldsymbol{c}^+(\lambda) \end{pmatrix}, \qquad \boldsymbol{T}^-(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & 0 \\ \boldsymbol{b}^+(\lambda) & \mathbb{1} \end{pmatrix},$$
$$\boldsymbol{S}^-(\lambda) = \begin{pmatrix} \boldsymbol{c}^-(\lambda) & 0 \\ -\boldsymbol{d}^+(\lambda) & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{T}^+(\lambda) = \begin{pmatrix} \mathbb{1} & -\boldsymbol{b}^-(\lambda) \\ 0 & \boldsymbol{a}^-(\lambda) \end{pmatrix},$$

These triangular factors can be viewed also as generalized Gauss decompositions of $T(\lambda)$ and its inverse:

$$T(\lambda) = \boldsymbol{T}^-(\lambda)\hat{\boldsymbol{S}}^+(\lambda) = \boldsymbol{T}^+(\lambda)\hat{\boldsymbol{S}}^-(\lambda), \qquad \hat{T}(\lambda) = \boldsymbol{S}^+(\lambda)\hat{\boldsymbol{T}}^-(\lambda) = \boldsymbol{S}^-(\lambda)\hat{\boldsymbol{T}}^+(\lambda).$$

The relations between $\boldsymbol{c}^\pm(\lambda)$, $\boldsymbol{d}^\pm(\lambda)$ and $\boldsymbol{a}^\pm(\lambda)$, $\boldsymbol{b}^\pm(\lambda)$ ensure that the next ones become identities and :

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda), \qquad G_0(\lambda) = \hat{D}^-(\lambda)(\mathbb{1} + K^-(\lambda)),$$
$$\chi^-(x,\lambda) = \chi^+(x,\lambda)\hat{G}_0(\lambda), \qquad \hat{G}_0(\lambda) = \hat{D}^+(\lambda)(\mathbb{1} - K^+(\lambda)),$$

valid for $\lambda \in \mathbb{R}$, where

$$D^-(\lambda) = \left(\begin{array}{cc} \boldsymbol{c}^-(\lambda) & 0 \\ 0 & \boldsymbol{a}^-(\lambda) \end{array} \right), \qquad K^-(\lambda) = \left(\begin{array}{cc} 0 & \boldsymbol{d}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & 0 \end{array} \right),$$

$$D^+(\lambda) = \left(\begin{array}{cc} \boldsymbol{a}^+(\lambda) & 0 \\ 0 & \boldsymbol{c}^+(\lambda) \end{array} \right), \qquad K^+(\lambda) = \left(\begin{array}{cc} 0 & \boldsymbol{b}^-(\lambda) \\ \boldsymbol{d}^+(\lambda) & 0 \end{array} \right),$$

Obviously the block-diagonal factors $D^+(\lambda)$ and $D^-(\lambda)$ are matrix-valued analytic functions for $\lambda \in \Omega_1 \cup \Omega_3$ and $\lambda \in \Omega_2 \cup \Omega_4$ respectively. Another well known fact about the FAS $\chi^\pm(x,\lambda)$ concerns their asymptotic behavior for $\lambda \to \pm\infty$, namely:

$$\xi^\pm(x,\lambda) = \chi^\pm(x,\lambda)e^{i\lambda^2 Jx}, \qquad \lim_{\lambda\to\infty} \xi^\pm(x,\lambda) = \mathbb{1}.$$

On the real and imaginary axis $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$ are related by

$$\xi^+(x,\lambda) = \xi^-(x,\lambda)G(x,\lambda), \quad G(x,\lambda) = e^{-i\lambda^2 Jx}G_0(\lambda)e^{i\lambda^2 Jx}, \quad G_0(\lambda) = S^+(\lambda)\hat{S}^-(\lambda).$$

The function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of L.

Parametrization of Lax pairs

Here we will outline a natural parametrization of $U(x,t,\lambda)$ and $V(x,t,\lambda)$ in terms of the local coordinate $Q_1(x,t)$ on the co-adjoint orbit $\mathfrak{g}^{(1)}$. Below we will choose it in the form:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix},$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are generic $p \times q$ and $q \times p$ matrices. Following Drinfeld, Sokolov (1981) we also introduce the solution $\xi(x,t,\lambda)$ of a RHP with canonical normalization. Since $\xi(x,t,\lambda)$ must be an element of the corresponding Lie group we define it by

$$\xi(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t),$$

where $\mathcal{Q}(x,t,\lambda)$ is a formal series over the negative powers of λ whose coefficients Q_s take values in $\mathfrak{g}^{(0)}$ if s is even and in $\mathfrak{g}^{(1)}$ if s is odd.

Therefore the first few of these coefficients take the form:

$$Q_1(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix}, \quad Q_2(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{r} & 0 \\ 0 & \boldsymbol{s} \end{pmatrix}, \quad Q_3(x,t) = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{v} \\ -\boldsymbol{w} & 0 \end{pmatrix}.$$

With such choice for $\xi(x,t,\lambda)$ we obviously have

$$\lim_{\lambda\to\infty} \xi(x,t,\lambda) = \mathbb{1}$$

which provides the canonical normalization of the RHP. Besides we have requested that $\mathcal{Q}(x,t,\lambda)$ takes values in the Kac-Moody algebra determined by the grading; in other words $\mathcal{Q}(x,t,\lambda)$ satisfies

$$\mathcal{Q}(x,t,\lambda) = C_0 \mathcal{Q}(x,t,-\lambda) C_0^{-1}, \qquad C_0 = \exp(\pi i J).$$

Then we can introduce $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as the non-negative parts of Gelfand-Dikii (1980), Drinfeld-Sokolov (1981):

$$U(x,t,\lambda) = -\left(\lambda^a \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+, \; V(x,t,\lambda) = -\left(\lambda^b \xi(x,t,\lambda) J \xi^{-1}(x,t,\lambda)\right)_+,$$

where a and b can be any integers. For simplicity and definiteness we will fix up $a = 2$ and $b = 4$. The explicit calculation of $U(x,t,\lambda)$ and $V(x,t,\lambda)$ in terms of $Q_s(x,t)$ can be done using the well known formula

$$\xi(x,t,\lambda)J\xi^{-1}(x,t,\lambda) = J + \sum_{s=1}^{\infty} \frac{1}{s!}\mathrm{ad}\,_{\mathcal{Q}}^{s} J, \quad \mathrm{ad}\,_{\mathcal{Q}} J = [\mathcal{Q}, J], \quad \mathrm{ad}\,_{\mathcal{Q}}^{2} J = [\mathcal{Q}, [\mathcal{Q}, J]], \quad \ldots$$

In particular for $a = 2$ and $b = 4$ we have:

$$\begin{aligned} U(x,t,\lambda) &= -\left(\lambda^2 \xi J \hat{\xi}\right)_+ = -\lambda^2 J + \lambda Q(x,t) + U_2(x,t), \\ Q(x,t) &= -[Q_1, J] = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}, \\ U_2(x,t) &= -\frac{1}{2}[Q_1, [Q_1, J]] - [Q_2(x,t), J] = \frac{1}{2}\begin{pmatrix} \boldsymbol{qp} & 0 \\ 0 & -\boldsymbol{pq} \end{pmatrix}. \end{aligned}$$

Note that since $Q_2(x,t) \in \mathfrak{g}^{(0)}$ then $[Q_2(x,t), J] = 0$. Similarly

$$
\begin{aligned}
V(x,t,\lambda) &= -\left(\lambda^4 \xi^{\pm} J \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ \\
&\quad V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 Q(x,t) - \lambda^4 J, \\
V_2(x,t) &= U_2(x,t), \qquad V_3(x,t) = -\frac{1}{2}\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} J - \frac{1}{6}\mathrm{ad}_{Q_1}^3 J, \\
V_4(x,t) &= -\frac{1}{2}\left(\mathrm{ad}_{Q_3}\mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_3} J\right) - \frac{1}{6}\left(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} J + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1}^2 J\right) \\
&\quad - \frac{1}{24}\mathrm{ad}_{Q_1}^4 J.
\end{aligned}
$$

Here we used again $[Q_2(x,t), J] = 0$ and $[Q_4(x,t), J] = 0$. Below we will pay special attention to the particular case $p = 1$ which corresponds to the vector GI equation.

RHP and multi-component GI equations

Here we assume that the FAS of L and M satisfy a canonical RHP with special reduction:

$$\xi^{\pm}(x,t,-\lambda) = \xi^{\pm,-1}(x,t,\lambda),$$

i.e., $\mathcal{Q}(x,t,\lambda) = -\mathcal{Q}(x,t,-\lambda)$ and therefore $Q_{2s}(x,t) = 0$. As a result the expressions for the Lax pair simplifies to

$$L\psi \equiv i\frac{\partial\psi}{\partial x} + U(x,t,\lambda)\psi = 0, \qquad M\psi \equiv i\frac{\partial\psi}{\partial t} + V(x,t,\lambda)\psi = 0,$$

$$U(x,t,\lambda) = U_2(x,t) + \lambda Q(x,t) - \lambda^2 J, \qquad Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix},$$

$$U_2(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{qp} & 0 \\ 0 & -\boldsymbol{pq} \end{pmatrix}, \qquad V_2(x,t) = U_2(x,t),$$

$$V_3(x,t) = \begin{pmatrix} 0 & \boldsymbol{v} - \frac{1}{6}\boldsymbol{qpq} \\ \boldsymbol{w} - \frac{1}{6}\boldsymbol{pqp} & \end{pmatrix},$$

where

$$V_4(x,t) = \frac{1}{2}\begin{pmatrix} \boldsymbol{q}\boldsymbol{w} + \boldsymbol{v}\boldsymbol{p} - \frac{1}{12}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p} & 0 \\ 0 & -\boldsymbol{w}\boldsymbol{q} - \boldsymbol{p}\boldsymbol{v} + \frac{1}{12}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q} \end{pmatrix}.$$

The commutation $[L, M]$ must vanish identically with respect to λ. It is polynomial in λ with the following coefficients:

$$\begin{aligned}
\lambda^5 :& \quad -[J, V_1] - [Q, J] = 0, && \Rightarrow \quad V_1 = Q, \\
\lambda^4 :& \quad -[J, V_2] + [Q, V_1] - [U_2, J] = 0, && \Rightarrow \quad \text{identity} \\
\lambda^3 :& \quad i\frac{\partial V_1}{\partial x} + [U_2, V_1] + [Q, V_2] = [J, V_3], &&
\end{aligned}$$

The last of these equations is fulfilled iff

$$\boldsymbol{v} = \frac{i}{2}\frac{\partial \boldsymbol{q}}{\partial x} + \frac{1}{6}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}, \qquad \boldsymbol{w} = -\frac{i}{2}\frac{\partial \boldsymbol{p}}{\partial x} + \frac{1}{6}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}.$$

The next equations are:

$$\begin{aligned}
\lambda^2: &\qquad i\frac{\partial V_2}{\partial x} + [U_2, V_2] + [Q, V_3] = [J, V_4] \equiv 0, \\
\lambda^1: &\qquad i\frac{\partial V_3}{\partial x} - i\frac{\partial Q}{\partial t} + [U_2, V_3] + [Q, V_4] = 0, \\
\lambda^0: &\qquad i\frac{\partial V_4}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_4] = 0.
\end{aligned}$$

The first of the above equations is satisfied identically. The second one written in block-components gives the following NLEE which can be viewed as multicomponent GI eqs. related to the **D**.III symmetric space:

$$\begin{aligned}
i\frac{\partial \boldsymbol{q}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{q}}{\partial x^2} - \frac{i}{2}\boldsymbol{q}\frac{\partial \boldsymbol{p}}{\partial x}\boldsymbol{q} + \frac{1}{4}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q} = 0, \\
-i\frac{\partial \boldsymbol{p}}{\partial t} + \frac{1}{2}\frac{\partial^2 \boldsymbol{p}}{\partial x^2} + \frac{i}{2}\boldsymbol{p}\frac{\partial \boldsymbol{q}}{\partial x}\boldsymbol{p} + \frac{1}{4}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p}\boldsymbol{q}\boldsymbol{p} = 0.
\end{aligned}$$

The last equation is a consequence of the expressions for Q_0 and V_4.

Soliton solutions

Dressing method

Zakharov-Shabat's dressing method (1974), (1978), (1980) (Zakharov - Mikhailov), VG., Grahovski, Valchev (2007). Construct new FAS $\chi^{\pm}(x,t,\lambda)$ from the known (bare) FAS, $\chi_0^{\pm}(x,t,\lambda)$ by the means of the so-called dressing factor $u(x,t,\lambda)$:

$$\chi^{\pm}(x,t,\lambda) = u(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda).$$

The dressing factor is analytic in the entire complex λ-plane, with the exception of the newly added simple pole singularities at $\lambda = \lambda_k^{\pm}$, $k = 1, 2, \ldots, N$:. It is known that these singularities are in fact discrete eigenvalues of the 'dressed' Lax operator L:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_k(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_k(x,t) \right).$$

As far as the FAS satisfy the Lax pair equations, the dressing factor must be a solution of the equation

$$iu_x + U_2 u - u U_2^{(0)} + \lambda(Qu - uQ^{(0)}) + \lambda^2 [u, J] = 0,$$

where the upper index (0) indicates the quantities, associated to the bare solution. The equation must hold identically with respect to λ. Since u has poles at finitely many points of the discrete spectrum, it will be enough to request that the equation holds for $\lambda \to \infty$ and $\lambda \to \lambda_k^{\pm}$. For $\lambda \to \infty$, $u \to \mathbb{1}$, so the derivative term disappears. The λ^2- terms are proportional to $[J, \mathbb{1}]$ that also identically vanishes. Thus, we are left with two terms, which are easily evaluated to be

$$\lambda^1 : \qquad Q - Q^{(0)} = \sum_{k=1}^{N} (\lambda_k^+ - \lambda_k^-)[J, B_k - \tilde{B}_k],$$

$$\lambda^0 : \qquad U_2 - U_2^{(0)} = \sum_{k=1}^{N} (\lambda_k^+ - \lambda_k^-) \Big([J, \lambda_k^+ B_k - \lambda_k^- \tilde{B}_k] - Q(B_k - \tilde{B}_k) + (B_k - \tilde{B}_k) Q^{(0)} \Big) .$$

Thus, if we know the residues $B_k, \tilde{B}_k$ we are able to reconstruct $Q(x,t)$ and $U_2(x,t)$. The condition holds for $\lambda \to \lambda_k^{\pm}$ leads to the following:

$$i\partial_x B_k + (U_2 + \lambda_k^+ Q)B_k - B_k(U_2^{(0)} + \lambda_k^+ Q^{(0)}) + (\lambda_k^+)^2 [B_k, J] = 0.$$

In the simplest possible nontrivial case, B_k are rank 1 matrices of the form

$$B_k = |n_k\rangle\langle m_k|$$

satisfying the matrix equation ($|n\rangle$ is a vector-column, $\langle m|$ is a vector-row as usual). It is straightforward to verify that B_k will satisfy the equation if and only if

$$i\partial_x |n_k\rangle + \left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J\right)|n_k\rangle = 0,$$
$$i\partial_x \langle m_k| - \langle m_k|\left(U_2^{(0)} + \lambda_k^+ Q^{(0)} - (\lambda_k^+)^2 J\right) = 0,$$

i.e.

$$|n_k\rangle = \chi^+(x,t,\lambda_k^+)|n_{k,0}\rangle, \qquad \langle m_k| = \langle m_{k,0}|\hat{\chi}_0^+(x,t,\lambda_k^+),$$

where $|n_{k,0}\rangle$ and $\langle m_{k,0}|$ are some constant vectors. One can start with the trivial bare solutions $Q^{(0)} = 0$, $U_2^{(0)} = 0$, so that $\chi_0^+(x,t,\lambda) = \exp i(\lambda^2 Jx + \lambda^4 Jt)$ is known explicitly.

Example - One soliton solution with local reduction

In the first example the dressing factor $u(x,\lambda;t)$ satisfies the reduction conditions from the first reduction of Mikhailov:

$$\text{A)} \qquad A_1 u^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = u^{-1}(x,t,\lambda), \qquad \text{B)} \qquad u(x,t,-\lambda) = u^{-1}(x,t,\lambda).$$

We consider the case $p = 1$, i.e. $\boldsymbol{q}$ is a vector-row and $\boldsymbol{p}$ is a vector-column, J is diagonal with $J_{11} = 1$ and $J_{ii} = -1$ for $i = 2, \dots n$. $(A_1)_{ij} = \epsilon_i \delta_{ij}$ is diagonal, with $\epsilon_i = \pm 1$. Introducing the notation

$$A_1 = \mathrm{diag}(a_1, a_2)$$

for the block-diagonal matrix A_1 and noting that $A_1 = A_1^{-1}$, we have the following relations between $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{q} = \kappa_1 a_1 \boldsymbol{p}^\dagger a_2, \qquad \boldsymbol{p} = \kappa_1 a_2 \boldsymbol{q}^\dagger a_1.$$

A dressing factor with simple poles at $\lambda = \lambda_1^\pm$ has the form

$$u(x,t,\lambda) = \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_1(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_1(x,t)$$

Moreover, the reductions A and B are simultaneously satisfied if

$$\lambda_1^+ = -\kappa_1 (\lambda_1^-)^*, \qquad \tilde{B}_1 = A_1 B_1^\dagger A_1^{-1}.$$

Let us introduce the notation $\mu \equiv \lambda_1^+$ and in polar form $\mu = \rho e^{i\varphi}$. Both reductions A, B must hold identically with respect to λ which necessitates (e.g. when $\lambda \to \mu$)

$$B_1 \left(\mathbb{1} - \frac{\mu + \kappa_1 \mu^*}{2\mu} B_1(x,t) + \frac{\mu + \kappa_1 \mu^*}{\mu - \kappa_1 \mu^*} A_1 B_1^\dagger(x,t) A_1^{-1} \right) = 0$$

Looking for a rank one solution $B_1 = |n\rangle\langle m|$ of the matrix equation ($|n\rangle$ is a vector-column, $\langle m|$ is a vector-row as usual) we find that

$$B_1 = z\frac{A_1|m^*\rangle\langle m|}{\langle m|A_1|m^*\rangle}$$

where the complex constant z satisfies the linear equation

$$1 - \frac{\mu + \kappa_1\mu^*}{2\mu}z + \frac{\mu + \kappa_1\mu^*}{\mu - \kappa_1\mu^*}z^* = 0.$$

In addition, from it follows that

$$i\partial_x B_1 + (U_2 + \mu Q)B_1 - B_1(U_2^{(0)} + \mu Q^{(0)}) + \mu^2[B_1, J] = 0,$$
$$Q = Q^{(0)} + (\mu + \kappa_1\mu^*)[J, B_1 - C_1 B_1^\dagger(x,t)C_1^{-1}].$$

and together with the assumption $B_1 = |n\rangle\langle m|$ one can find out that $\langle m|$ satisfies the bare equation

$$i\partial_x\langle m| - \langle m|(U_2^{(0)} + \mu Q^{(0)} - \mu^2 J) = 0.$$

Therefore, starting from the trivial solution $U_2^{(0)} = Q^{(0)} = 0$ we find

$$\langle m| = \langle m_0| e^{i(\mu^2 x + \mu^4 t)J},$$

where $\langle m_0|$ is a constant vector with components m_{0j}. Now we can write the one-soliton solution,

$$\boldsymbol{q}_{j-1}(x,t) = Q_{1j} = 4\rho r(\kappa_1) \frac{m_{0j} e^{\xi_0} e^{-i\phi(x,t)}}{m_{01} \cosh(\theta(x,t) - \xi_0)}, \qquad j = 2, \ldots, n,$$

where $r(1) = i \sin\varphi$, and $r(-1) = \cos\varphi$ and when $A_1 = \mathbb{1}$,

$$e^{-2\xi_0} \equiv \frac{\sum_{j=2}^{n} |m_{0j}|^2}{|m_{01}|^2}$$

is real and positive,

$$\theta(x,t) = 2\rho^2 (\sin 2\varphi) x + 2\rho^4 (\sin 4\varphi) t, \qquad \phi(x,t) = 2\rho^2 (\cos 2\varphi) x + 2\rho^4 (\cos 4\varphi) t,$$

Example - One soliton solution with nonlocal reduction

In the second example the dressing factor $u(x,t,\lambda;t)$ satisfies the reduction conditions from the first reduction of Mikhailov:

$$\text{A)}\quad C_1 u^\dagger(-x,-t,\kappa_1\lambda^*)C_1^{-1} = u^{-1}(x,\lambda), \qquad \text{B)}\quad u(x,t,-\lambda) = u^{-1}(x,\lambda).$$

Let us take for simplicity $p = 1$, $n = 2$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are scalar functions. The automorphism C_1 can not be represented by a diagonal matrix, since now it must change the sign of $J \equiv \sigma_3$. Hence, we take

$$C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The reduction gives now the following connections between $\boldsymbol{p}$ and $\boldsymbol{q}$, under which the equations are $\mathcal{CPT}$-invariant:

$$\boldsymbol{q}(x,t) = -\kappa_1 \boldsymbol{q}^*(-x,-t), \qquad \boldsymbol{p}(x,t) = -\kappa_1 \boldsymbol{p}^*(-x,-t).$$

The dressing factor satisfies the equation. Again it is taken to have simple poles at $\lambda = \lambda_1^{\pm}$:

$$u(x,t,\lambda) = \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} B_1(x,t) + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} \tilde{B}_1(x,t)$$

This time the reductions A and B are simultaneously satisfied if

$$\lambda_1^+ = -\kappa_1 (\lambda_1^-)^*, \qquad \tilde{B}_1(x,t) = C_1 B_1^\dagger(-x,-t) C_1^{-1}.$$

With the short notations $\mu \equiv \lambda_1^+ = \rho e^{i\varphi}$ we obtain the equation for $B_1(x,t)$

$$B_1(x,t) \left(\mathbb{1} - \frac{\mu + \kappa_1 \mu^*}{2\mu} B_1(x,t) + \frac{\mu + \kappa_1 \mu^*}{\mu - \kappa_1 \mu^*} C_1 B_1^\dagger(-x,-t) C_1^{-1} \right) = 0$$

with a rank one solution

$$B_1(x,t) = z \frac{C_1 |m^*(-x,-t)\rangle \langle m(x,t)|}{\langle m(x,t)| C_1 | m^*(-x,-t)\rangle}$$

where the complex constant z and the components of $\langle m(x,t)|$, i.e. $m_j(x,t)$ are as before. The solution is

$$Q(x,t) = Q^{(0)}(x,t) + (\mu + \kappa_1\mu^*)[J, B_1(x,t) - C_1 B_1^\dagger(-x,-t)C_1^{-1}].$$

Starting with $Q^{(0)}(x,t) \equiv 0$ and real m_{0j} we obtain

$$\boldsymbol{q}(x,t) = Q_{12} = (\mu + \kappa_1\mu^*)(z - z^*)\frac{m_{02}e^{-i\phi(x,t)}}{m_{01}\cosh(\theta(x,t))},$$

$$\boldsymbol{p}(x,t) = Q_{21} = -(\mu + \kappa_1\mu^*)(z - z^*)\frac{m_{01}e^{i\phi(x,t)}}{m_{02}\cosh(\theta(x,t))},$$

with $\phi(x,t)$ and $\theta(x,t)$ defined as before,

$$(z - z^*)_{\kappa_1=1} = 2i\tan\varphi, \qquad (z - z^*)_{\kappa_1=-1} = -2i\cot\varphi.$$

It is worth noting that in both cases the action of the reduction on λ is $\lambda \to \epsilon\lambda^*$. In both cases the action on λ is very nice. Indeed, the analyticity regions are $A_+ = \operatorname{Im}\lambda^2 > 0$ and $A_- = \operatorname{Im}\lambda^2 < 0$. The action on λ always maps $A_+ \to A_-$.

Integrals of motion of the multi-component DNLS equations

We conclude that block-diagonal Gauss factors $D_J^{\pm}(\lambda)$ are generating functionals of the integrals of motion. The principal series of integrals is generated by $m_1^{\pm}(\lambda)$:

$$\pm \ln m_1^{\pm} = \frac{1}{i} \sum_{s=1}^{\infty} I_s \lambda^{-s}.$$

Let us calculate their densities as functionals of $Q(x,t)$ – use of the third type of Wronskian identities:

$$\left(i\hat{\xi}^{\pm}\dot{\xi}^{\pm}(x,\lambda) + 2\lambda J x \right)\Big|_{x=-\infty}^{\infty} = -\int_{-\infty}^{\infty} dx\, \left(\hat{\xi}(Q(x) - 2\lambda J)\xi(x,\lambda) + \lambda^2 [J, \hat{\xi}^{\pm}\dot{\xi}^{\pm}(x,\lambda)] \right),$$

Multiply both sides with J and take the Killing form:

$$\left\langle (i\hat{\xi}^{\pm}\dot{\xi}^{\pm}(x,\lambda) + 2\lambda J x, J \right\rangle\Big|_{x=-\infty}^{\infty} = \pm 2i \frac{d}{d\lambda} \ln m_1^{\pm}(\lambda),$$

which means that

$$\pm i \frac{d}{d\lambda} \ln m_1^{\pm}(\lambda) = \frac{i}{2} \int_{-\infty}^{\infty} dx \, \Big(\langle (Q(x) - 2\lambda J), \xi^{\pm}(x,\lambda) J \hat{\xi}^{\pm}(x,\lambda) \rangle + 2\lambda \langle J, J \rangle \Big) \, .$$

If we introduce the notations:

$$\xi^{\pm} J \hat{\xi}^{\pm}(x,\lambda) = J + \sum_{s=1}^{\infty} \lambda^{-s} X_s,$$

then we can calculate recursively X_s. Knowing X_s we find recursive formula for I_s:

$$I_{2s} = \frac{1}{4s} \int_{-\infty}^{\infty} dx \, (\langle Q(x), X_{2s+1} \rangle - 2 \langle J, X_{2s+2} \rangle) \, .$$

In our case $Q_2 = Q_4 = \cdots = 0$, $X_{2s} \in \mathfrak{g}^{(0)}$, $X_{2s+1} \in \mathfrak{g}^{(1)}$. Therefore $I_1 = I_3 = \cdots = 0$ and

$$I_1 = 0, \qquad I_2 = \frac{1}{4} \int_{-\infty}^{\infty} dx \, (i \langle \boldsymbol{q}_x, \boldsymbol{p} \rangle - i \langle \boldsymbol{q}, \boldsymbol{p}_x \rangle + \langle \boldsymbol{qp}, \boldsymbol{qp} \rangle) \, ,$$

$$I_3 = 0, \qquad I_4 = \frac{1}{4} \int_{-\infty}^{\infty} dx \left(\langle \boldsymbol{q}_x, \boldsymbol{p}_x \rangle + \frac{i}{2} (\langle \boldsymbol{q}_x, \boldsymbol{pqp} \rangle - \langle \boldsymbol{qpq}, \boldsymbol{p}_x \rangle) + \frac{1}{4} \langle \boldsymbol{qpq}, \boldsymbol{pqp} \rangle \right) .$$

Conclusions

- The multi-component Kaup-Newell and GI hierarchies on symmetric spaces, and their hierarchy of Hamiltonian structures are constructed.

  The results of this paper can be extended in several directions:

- To study the gauge equivalent systems to the multi-component KN and GI equations on symmetric spaces.

- To extend our results for the case of non-vanishing boundary conditions

$$\lim_{x \to \pm\infty} Q(x,t) = Q_{\pm}. \qquad (Q_+)^2 = (Q_-)^2.$$

  This condition ensures that the spectra of the asymptotic operators $L_{\pm}$ coincide.

- To study quadratic bundles associated with other types Hermitian symmetric spaces both for Kaup-Newell and for GI equations.