Singular sets of Yang-Mills minimizers in dimension higher than 4

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The Plateau Problem

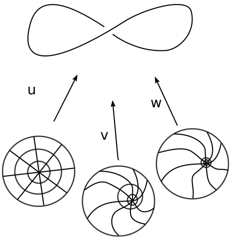
Question: Which is the surface $\Sigma \subset \mathbb{R}^3$ of smallest area with a fixed boundary γ?

Parametric approach [Douglas-Radó]

- $u : D^2 \to \mathbb{R}^3$ immersion,
  $u|_{\partial D^2}$ parameterization of γ.
- Minimize

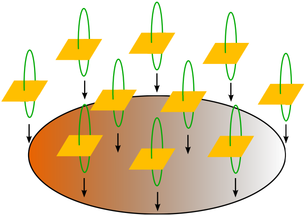
$$A(u) := \int_{D^2} |\partial_x u \times \partial_y u|.$$

- Problem:
  Large invariance group $\mathit{Diff}^+(D^2)$.



A NON-INTEGRABLE PLATEAU PROBLEM

- M compact Riemannian manifold, $P \to M$ "principal G bundle" with a connection ($\sim$ horizontal distribution Q).
- Fix $Q|_{\partial M}$.
- Question: Which is the best extension of Q inside M? Find the "most integrable" Q.



Frobenius: Q integrable $\Leftrightarrow$ $[X, Y] \in Q$ for all $X, Y \in Q$.

$$\text{Minimize } \int_M \sum_{i,j} |\mathrm{Vert}^Q([\tilde{X}_i, \tilde{X}_j])|^2 = \int_M |F|^2.$$

The Yang-Mills Lagrangian

A is a 1-form with coeff. in the Lie algebra $\mathfrak{g}$ of G.

(e.g. $G = SU(2), \mathfrak{g} = \mathfrak{su}(2)$) $Q \sim \nabla \overset{loc}{=} d + A \sim$ connection.

$$\mathcal{Y}M(A) = \int_B |F_A|^2 dx = \int_B |dA + A \wedge A|^2 dx,$$

- Yang-Mills fields are critical points under perturbation $A + ta$
- Yang-Mills equation: $D_A^* F_A = 0$ (recall Bianchi identity $D_A F_A = 0$)
- in coordinates:

$$\sum_i \partial_{x_i} F_{ij} + [A_i, F_{ij}] = 0 \quad \text{Bianchi:} \sum_{perm\ i,j,k} \partial_{x_k} F_{ij} + [A_k, F_{ij}] = 0 \ .$$

- $G = U(1) = \{e^{i\theta}\}$: electromagnetism, $\mathfrak{g} \simeq \mathbb{R}, D_A = d, D_A^* = d^*$ and $F_A = dA$. $d^* F_A = 0, dF_A = 0$ become $curl\vec{F} = 0, div\vec{F} = 0$.

Why to study Yang-Mills theory at very low regularity?

- 4D theory brought groundbreaking results (e.g. Donaldson '84)
- 4D constructions: direct minimization (Uhlenbeck '82, Sedlacek '82), algebraic constructions (Atyiah-Hitchin-Drinfeld-Manin '78), gluing (Taubes)
- Geometric program by Tian '00, Donaldson-Segal '11 for gauge theory in higher dimension.
- Algebraic or gluing construction for Yang-Mills connections in dim. $n \geq 5$?

Compactness and regularity

- Like in the parametric Plateau, gauge invariance. For $g : B^n \to G$ we have

$$A_i^g = g^{-1}\partial_{x_i} g + g^{-1} A_i g \,, \quad F_{ij}^g := g^{-1} F_{ij} g \,, \quad |g^{-1} F g| = |F| \,.$$

- Large invariance group ... How to break the gauge?
- Inspired by the abelian case, look after Coulomb condition $\sum_i \partial_{x_i} A_i = 0$

$$\text{i.e. solve:} \quad \operatorname{div}(g^{-1}\nabla g) = -\operatorname{div}(g^{-1}\vec{A} g) \,.$$

$$\text{Yang-Mills:} \quad \forall j \; \Delta A_j^g = \sum_i [A_i^g, \partial_{x_j} A_i^g] - [A_i^g, [A_i^g, A_j^g]] \,.$$

In these coordinates the equations are elliptic quasilinear, critical in 4D.

A NAÏVE APPROACH:

$$\min_{g\in W^{1,2}(B,G)} \int_B |g^{-1}\nabla g + g^{-1}\vec{A}g|^2 dx\ .$$

- Problem 1: we get a minimizer $A^g \in L^2$ and not in $W^{1,2}$: bad for regularity
- Problem 2: A^g is not controlled by $\|F\|_{L^2}$, but only by A: bad for compactness

Theorem (Uhlenbeck '82)

Let $n \leq 4$. Then $\exists \epsilon_0 > 0$ such that if $\|dA + A\wedge A\|_{L^2} \leq \epsilon_0$ and $A \in W^{1,2}$ then $\exists$ gauge $g \in W^{2,2}$ such that

$$\|A^g\|_{W^{1,2}} \leq C\|dA + A\wedge A\|_{L^2}, \quad \text{and} \quad d^*A^g = 0.$$

MINIMIZATION IN 4D

The correct space where to study the Yang-Mills Lagrangian in 4D is given by Sobolev connections:

$$\mathcal{A}^{1,2}(M) := \left\{ \begin{array}{l} A \in L^2(M) \text{ s.t. } YM(A) < \infty \text{ and} \\ \text{locally } \exists g : U \to G, \quad A^g \in W^{1,2} \end{array} \right\} .$$

- Let $A_k \in \mathcal{A}^{1,2}(E)$, $F_k \overset{L^2}{\rightharpoonup} F$, $\sup_k \|F_k\|_{L^2} \leq C$.

  Then up to subsequence $A_k \overset{W^{1,2}_{loc}}{\rightharpoonup} A$ outside finitely many points.
- $A \in \mathcal{A}^{1,2}(\tilde{E})$ (Uhlenbeck, point removability)

Supercritical dimensions

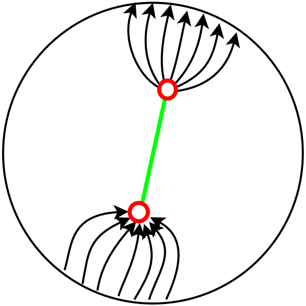
- The limit connection might not be $W^{1,2}_{loc}$ anymore in dimension 5!
- Fact: $A^g \in W^{1,2}_{glob}(S^4)$; $A^g \in W^{1,2}_{glob}(\partial B^5_r)$ then the bundle is trivial:

$$tr(F \wedge F) = d\left[tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right].$$

$$\frac{1}{8\pi^2}\int_{S^4} tr(F \wedge F) = c_2(E) \in \mathbb{Z} \quad \text{Chern number}$$

SUPERCRITICAL DIMENSIONS

- Loss of $W^{1,2}$-regularity for $n = 5$: [Tian '00, Tao-Tian '04]
- $X_k := *tr(F_k \wedge F_k)$ with $F_k \overset{L^2}{\rightharpoonup} F$.



- Chern classes are $\neq 0$ over $\partial B_r(x^{\pm})$ for all $r > 0$.
- What is the correct replacement for $\mathcal{A}^{1,2}$?

Weak connections and closure results

[P.- Rivière, '13]: definition of space $\mathcal{A}_G(\mathbb{B}^5)$ of weak connections.

Theorem (P.- Rivière, '13)

Let $A_k \in \mathcal{A}_G(\mathbb{B}^5)$ such that $\|F_k\|_{L^2(\mathbb{B}^5)} \leq C$ and F_k weak-L^2 converge to F. Then $F = dA + A \wedge A$, $A \in \mathcal{A}_G(\mathbb{B}^5)$ as well.

$$\mathcal{A}_G(\mathbb{B}^5) := \left\{ \begin{array}{c} A \in L^2,\ F_A \overset{\mathcal{D}'}{=} dA + A \wedge A \in L^2 \\ \forall p \in \mathbb{B}^5 \text{ a.e. } r > 0,\ \exists A(r) \in W^{1,2}_{loc}(\partial B_r(p)) \\ i^*_{\partial B_r(p)} A \sim A(r) \end{array} \right\} .$$

WEAK CONNECTIONS AND CLOSURE RESULTS

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Ingredients of the proof:

- Geometric “distance” on slices: $\mathrm{dist}(A, B) = \inf\{\|A - B^g\|_{L^2(S^4)} \,|\, g \in \mathcal{G}\}$.
- Oscillation control on slices: $\mathrm{dist}(A(t), A(t')) \leq C\|F\|_{L^2(\mathbb{B}^5)}|t - t'|^{1/2}$.
- Sublevels of $\|F\|_{L^2(S^4)}$ are sequentially compact for dist.
- MBV-compactness type result.

SLICING METHODS

Theorem (Federer-Fleming, '60)

Let $I_j \rightharpoonup I$ be a weakly convergent sequence of integral k-currents such that

$$\sup_j \mathbb{M}(I_j) \leq C, \quad \sup_j \mathbb{M}(\partial I_j) \leq C.$$

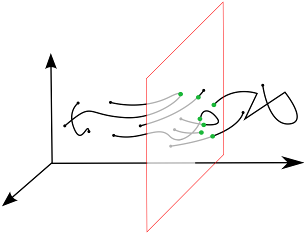
Then I is an integral current as well.

Qualitative⇝Quantitative
Closure⇝Compactness

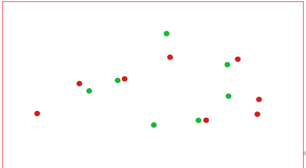
- Jerrard '02, White '99 relation to the compactness of *BV* functions
- Ambrosio-Kirchheim '00 extend the closure theorem to metric currents
- Hardt-Rivière '03 introduce the notion of scans De Pauw-Hardt

Slicing and scans

- I integral 1-current in $\mathbb{R}^3$.
  Slices by parallel planes:



- We obtain $f : \mathbb{R} \to (I_0(\mathbb{R}^2), d_{flat})$ of *metric bounded variation*.
- $d_{flat}(X, Y) := \inf\{\mathbb{M}(A) + \mathbb{M}(B) : X - Y = \partial A + B\}$



Regularity of minimizing weak curvatures

Yang-Mills minimizers in 5D have isolated singular points:

Theorem (P.- Rivière, '13)

Let $A \in \mathcal{A}_G^{\phi}(\mathbb{B}^5)$ be a minimizer (or stationary critical point) for the Yang-Mills-Plateau problem. Then A corresponds to a smooth connection over a classical bundle $E \to \mathbb{B}^5 \setminus S$, where S is a set of isolated points.

- [Tian, '00], [Tao-Tian, '04] considered *admissible* Yang-Mills connections (smooth outside a compact 1-rectifiable set): smaller class, with no weak closure $\Rightarrow$ no existence of minimizers.

Ingredients:

- Approximation by $\mathcal{R}_G^{\infty}(\mathbb{B}^5)$, i.e. connections with isolated defects.
- Uhlenbeck ϵ-regularity in Morrey norms [Meyer-Rivière, '03], [Tao-Tian, '04].

Morrey approximability and regularity

Theorem (Meyer- Rivière '03, Tao-Tian '04)
If A, F are approximable by smooth A_i, F_i in $L^2(B^n)$ norm and

$$\sup_{x,r} \frac{1}{r^{n-4}} \int_{B_r(x)} |F_i|^2 < \epsilon_1 \ ,$$

then $\exists\, g \in W^{1,2}(\mathbb{B}^n, G)$ *such that* $A^g = g^{-1}dg + g^{-1}Ag$ *satisfies*

- $d^*A^g = 0$ *on* $\mathbb{B}^5$,
- $$\left(\sup_{x,r} \frac{1}{r^{n-4}} \int_{B_r(x)} |A^g|^4\right)^{\frac{1}{4}} + \left(\sup_{x,r} \frac{1}{r^{n-4}} \int_{B_r(x)} |DA^g|^2\right)^{\frac{1}{2}} \leq C\|F\|_{M(\mathbb{B}^5)} \ .$$

Approximation of weak curvatures

$$\mathcal{R}^{\infty,\phi}(\mathbb{B}^5) := \left\{ \begin{array}{c} F \text{ corresponding to some } [A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5) \text{ s.t.} \\ \exists k, \exists a_1, \ldots, a_k \in \mathbb{B}^5, \quad F = F_\nabla \text{ for a smooth connection} \nabla \\ \text{on some smooth } G\text{-bundle } E \to \mathbb{B}^5 \setminus \{a_1, \ldots, a_k\} \end{array} \right\}$$

Theorem (P.- Rivière, '13)

Let $[A] \in \mathcal{A}_G^{\phi}(\mathbb{B}^5)$, $F \in L^2(\mathbb{B}^5)$ curvature form corresp. to $A \in [A]$. Then there exist $F_k \in \mathcal{R}^{\infty,\phi}(\mathbb{B}^5)$ such that

$$A_k \xrightarrow{L^2} A\,, \quad F_k \xrightarrow{L^2} F.$$

If $\|F\|_M \leq \epsilon_0$ then we may find A_k, F_k locally smooth with $\|F_k\|_M \leq \epsilon_1$.

Strategy like for harmonic maps: [Bethuel, '91] did $W^{1,p}(\mathbb{B}^{n+1}, \mathbb{S}^n)$ instead of $\mathcal{A}_G(\mathbb{B}^5)$.

THE PROOF OF PARTIAL REGULARITY

- If $\|F\|^2_{M(\mathbb{B}^5)} < \epsilon_1$ then up to rescaling, for "good grid" $\int_{\mathbb{S}^4} |F|^2 < \epsilon_1$
- We then extend smoothly even on "bad" and obtain F_i, A_i as needed for the Morrey-Uhlenbeck ϵ-regularity.
- Obtain Coulomb gauge A^g with Morrey control and satisfying Yang-Mills' equation $\Delta A^g = -d^*[A^g, A^g] - *[A^g, *F_{A^g}]$.
- Apply regularity theory from [Meyer- Rivière '03]: A, F *approximable, stationary Yang-Mills,* $\frac{1}{r}\int_{B_r(x)} |F|^2 < \epsilon$ *then* A *smooth on* $B_{r/2}(x)$.
- Dimension reduction gives the result.

Relation with coherent reflexive sheaves

Algebraic singularities: $\mathcal{S}$ is a *coherent* sheaf of $\mathcal{O}$-modules if

1. (analytic structure) $\mathcal{O}_U^p \to \mathcal{S}_U \to 0$
2. (finiteness) $\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{S}_U \to 0$.

$\mathcal{S}$ is *reflexive* if "$\mathcal{S} = \mathcal{S}^{**}$". *Singular set* $\mathrm{sing}(\mathcal{S}) := \{x \in M : \mathcal{S}_x \text{ not free}\}$

Theorem

$\mathcal{S}$ reflexive coherent sheaf over M^n then $\dim_{\mathbb{C}}(\mathrm{sing}(\mathcal{S})) \leq n-3$.

Conjecture (Tian '00)

(ω, M) Kähler manifold, $\dim_{\mathbb{C}} M = n$, F curvature form of a Hermitian vector bundle which is

- *compatible:* $F^{0,2} = 0$
- ω^{n-2}*-anti-selfdual:* $*(\omega^{n-2} \wedge F) = -F$

Then $\mathrm{sing}(F)$ rectifiable of $\dim_{\mathbb{C}} \leq n-3$ (and calibrated by ω).

(Next goal: Isolated singular points for compatible F in 6 dimensions.)

IS THE 5D RESULT OPTIMAL?

Theorem (P. '13)

In 5D, the radial instanton $A_{\mathbb{B}^5} := \left(\frac{x}{|x|}\right)^* A_{\mathbb{S}^4}$ *is energy-minimizing.*

- cfr. [Hardt-Lin-Wang '97] $x/|x| : \mathbb{B}^3 \to \mathbb{S}^2$ is a minimizing (p-)harmonic map
- In particular, conjecture of [Tian '00] needs the complex structure condition ($A_{\mathbb{B}^5}$ is dr-antiselfdual)
- For the proof, use the technique of [P. '13] giving the regularity for the abelian case.

Discretizing the problem

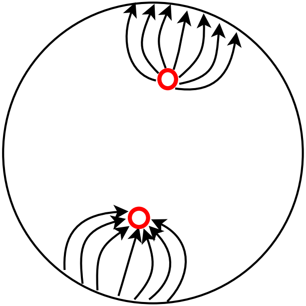
Theorem (Smirnov, '94)

Let X be an acyclic normal 1-current on $\mathbb{B}^3$. There exists a finite positive Borel measure μ on the space of arcs such that

$$\begin{aligned}
\langle X, \omega\rangle &= \int \langle [\gamma], \omega\rangle d\mu(\gamma), \\
\langle \|X\|, \phi\rangle &= \int \langle \|[\gamma]\|, \phi\rangle d\mu(\gamma), \\
\langle \partial X, f\rangle &= \int \langle \partial[\gamma], f\rangle d\mu(\gamma), \\
\langle \|\partial X\|, \phi\rangle &= \int \langle \|\partial[\gamma]\|, \phi\rangle d\mu(\gamma),
\end{aligned}$$

for all $\omega \in C^\infty(\mathbb{B}^3, \wedge^1\mathbb{R}^3), f \in C^\infty(\mathbb{B}^3), \phi \in C^0(\mathbb{B}^3)$.

- Represent X by its "flow-lines"

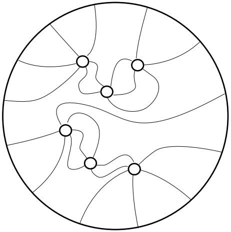


- Borel partition of arcs based on their endpoints

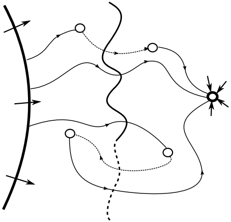
THE PROOF OF MINIMALITY

- Weak closure $\Rightarrow$ there exists a minimizer $[A] \in \mathcal{A}^{A_{\mathbb{S}^4}}_{SU(n)}(\mathbb{B}^5)$
- Use approximation result: $[A]$ is approximated by $[\tilde{A}] \in \mathcal{R}^{\infty}(\mathbb{B}^5)$
- Smirnov-decompose the vector field $X = *\mathrm{tr}(F_{\tilde{A}} \wedge F_{\tilde{A}})$
- Obtain combinatorial problem (on Borel sets of arcs)
- Find combinatorial competitor

THE COMBINATORIAL PICTURES



- Reduce to combinatorial problem
- Prove that there exists only one charge via Maxflow-Mincut



Similar methods give regularity in Abelian case (group $U(1)$) in

Some interesting questions/topics

1. Asymptotics of large number of interacting topological sigularities (Coulomb/Riesz gasses = linear model problem cf. [P-Serfaty '15], [P '15])

2. Extending $u \in W^{1,n}(\mathbb{S}^n, M)$ to $U \in W^{1,n+1}_{weak}(\mathbb{B}^{n+1}, M)$ by inserting topological singularities, with norm control cf. [P-Riviere '14], [P-Van Schaftingen, coming soon!]

3. Minimal connections between topological singularities and weak sequential approximability cf. [Bethuel '14], [P-Züst '14]

4. Relation to optimal transportation and extension of Smirnov's method cf. [P-Brasco '12], [P '13]

5. Use the approximation method to construct the Yang-Mills-Gibbs measure

$$d\mu_T([A]) = \frac{1}{Z_T} \exp\left(-\frac{1}{T}\mathcal{YM}(A)\right) D[A]$$

(presently known by probabilists in $2D$ only! cf. [Levy, Levy-Norris '00-'05])