# The Jacobi group in Physics and Mathematics

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*GAP@DFT, January 23, 2015*

# Outline

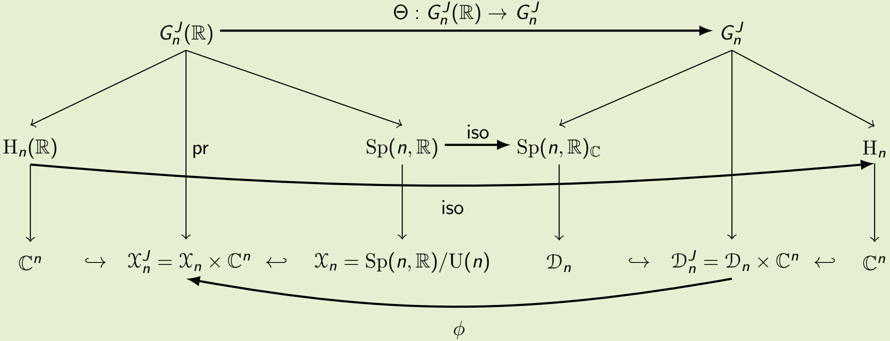
# Jacobi group $G_n^J$

• Jacobi group
$G_n^J(\mathbb{R}) = \mathrm{H}_n(\mathbb{R}) \rtimes \mathrm{Sp}(\mathrm{n}, \mathbb{R})$, $\mathrm{H}_n(\mathbb{R})$- Heisenberg
Siegel-Jacobi upper half plane: $\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n$, $\mathcal{X}_n = \mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$
Siegel upper half plane:
$\mathcal{X}_n = \{V \in M_n(\mathbb{C}) | V = S + \mathrm{i}R, V = V^t, S, R \in M_n(\mathbb{R}), R > 0\}$

• complex version:
$G_n^J = \mathrm{H}_n \rtimes \mathrm{Sp}(\mathrm{n}, \mathbb{R})_{\mathbb{C}}$, $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{U}(n, n)$
Siegel-Jacobi ball: $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$, $\mathcal{D}_n = \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}/\mathrm{U}(n)$
Siegel ball: $\mathcal{D}_n = \{W \in M_n(\mathbb{C}) | W = W^t, \mathbb{1}_n - W\bar{W} > 0\}$

Comment: in Math: M. Eichler & D. Zagier 1985; R. Berndt & R. Schmidt 1998; E. Kähler 1983, ..., 1992, K. Takase, J.-H. Yang...:+ Physics: U. Niederer 1972, C. R. Hagen 1972, K. B. Wolf ...

# Summary of notation



# Jacobi algebra "In Physics"

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \quad [\mathfrak{h}_1, \mathfrak{g}_1^J] = \mathfrak{h}_1$$

$$\begin{aligned}
&[a, a^\dagger] = 1, \\
&[K_0, K_\pm] = \pm K_\pm \ , \ [K_-, K_+] = 2K_0, \qquad * * \\
&[a, K_+] = a^\dagger \ , \ \left[K_-, a^\dagger\right] = a, \ \left[K_+, a^\dagger\right] = [K_-, a] = 0, \\
&\left[K_0, \ a^\dagger\right] = \frac{1}{2}a^\dagger, [K_0, a] = -\frac{1}{2}a.
\end{aligned}$$

# Jacobi group - in Math

The real Jacobi group $G_1^J(\mathbb{R})$ - subgroup of $Sp(2,\mathbb{R})$: $4\times 4$ real matrices $g=((\lambda,\mu,\kappa),M)$, $(\lambda,\mu,\kappa)\in H(\mathbb{R})$, $M\in SL(2,\mathbb{R})$

$$g=\left(\begin{array}{cccc} a & 0 & b & a\mu-b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu-d\lambda \\ 0 & 0 & 0 & 1 \end{array}\right),\; M=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right),\; ad-bc=1,$$

$\mathfrak{g}^J=\langle \mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathbb{R}}$, $\mathfrak{h}=\langle \mathrm{P},\mathrm{Q},\mathrm{R}\rangle_{\mathbb{R}}$, $\mathfrak{sl}(2,\mathbb{R})=\langle \mathrm{F},\mathrm{G},\mathrm{H}\rangle_{\mathbb{R}}$. P, Q, R, F, G, H are $4\times 4$ matrices of coefficients $\mathrm{F}_{ij}=\delta_{i1}\delta_{j3}$, $\mathrm{G}_{ij}=\delta_{i3}\delta_{j1}$, $\mathrm{H}_{ij}=\delta_{i1}\delta_{j1}-\delta_{i3}\delta_{j3}$, $\mathrm{P}_{ij}=\delta_{i2}\delta_{j1}-\delta_{i3}\delta_{j4}$, $\mathrm{Q}_{ij}=\delta_{i1}\delta_{j4}+\delta_{i2}\delta_{j3}$, $\mathrm{R}_{ij}=\delta_{i2}\delta_{j4}$, where $i,j=1,2,3,4$.

$$[\mathrm{P},\mathrm{Q}]=2\mathrm{R},\;\; [\mathrm{F},\mathrm{G}]=\mathrm{H},\;\; [\mathrm{H},\mathrm{F}]=2\mathrm{F},\;\; [\mathrm{G},\mathrm{H}]=2\mathrm{G},$$
$$[\mathrm{P},\mathrm{F}]=\mathrm{Q},\;\; [\mathrm{Q},\mathrm{G}]=\;\mathrm{P},\;\; [\mathrm{P},\mathrm{H}]=\;\mathrm{P},\;\; [\mathrm{H},\mathrm{Q}]=\;\;\mathrm{Q},$$

# Continuation

$$a = \frac{1}{2\sqrt{|\mu|}}\left(P - \mathrm{i}\sigma Q\right), \;\; a^{\dagger} = -\frac{1}{2\sqrt{|\mu|}}\left(P + \mathrm{i}\sigma Q\right),$$

$$K_{\pm} = \mp\frac{1}{2}H - \frac{\mathrm{i}\sigma}{2}\left(F + G\right), \;\; K_0 = \frac{\mathrm{i}\sigma}{2}\left(G - F\right),$$

$\sigma = \mu / |\mu|$, $\mu = 2\pi m$, $m \in \mathbb{R}$. In Q. M. $\hbar = (2\mu^2)^{-1}$.

## Proposition

$\hat{\pi}(\mathfrak{g}^J_{\mathbb{C}}) = \left\langle I, a, a^{\dagger}, K_+, K_-, K_0 \right\rangle_{\mathbb{C}}$ *with* $\left(a^{\dagger}\right)^{\dagger} = a$, $K^{\dagger}_{\pm} = K_{\mp}$ *and (**).*

# Coherent states based $\mathcal{D}_1^J$

$$e_{z,w} = \mathrm{e}^{z\mathbf{a}^{\dagger}+w\mathbf{K}_+}e_0,\ z,w\in\mathbb{C},\ |w|<1$$

$$\mathbf{a}e_0 = 0,\ \mathbf{K}_-e_0 = 0,\ \mathbf{K}_0e_0 = ke_0;\ k>0,$$

$e_0 = \varphi_0\otimes\phi_{k0}$. The Casimir operator $C = K_0^2 - K_1^2 - K_2^2$, $q = k(k-1)$; $k$ - the Bargmann index for the unitary irreducible positive discrete series representation $D_k^+$ of $\mathrm{Sp}(2,\mathbb{R})\approx \mathrm{SU}(1,1)$.

*Displacement operator*

$$D(\alpha) = \exp(\alpha\mathbf{a}^{\dagger}-\bar{\alpha}\mathbf{a}) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\mathbf{a}^{\dagger})\exp(-\bar{\alpha}\mathbf{a}),$$

*Unitary squeezed operator* of the $D_+^k$ representation of the group SU(1,1), $\underline{S}(z) = S(w)$ ($w = FC(z) = \frac{z}{|z|}\tanh(|z|),\ \eta = \ln(1-|w|^2)$):

$$\begin{aligned}\underline{S}(z) &= \exp(z\mathbf{K}_+ - \bar{z}\mathbf{K}_-),\ z\in\mathbb{C};\\ S(w) &= \exp(w\mathbf{K}_+)\exp(\eta\mathbf{K}_0)\exp(-\bar{w}\mathbf{K}_-),\ |w|<1.\end{aligned}$$

# Squeezed states

## Proposition

*FC-transform:* $(\alpha, w) \to (z, w)$

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0 = (1 - w\bar{w})^k \exp(-\frac{\bar{\alpha}}{2}z)e_{z,w}, z = \alpha - w\bar{\alpha}.$$

Instead of $e_0 = \varphi_0 \otimes \phi_{k0}$ , take $e_0 = \varphi_0$. This group admits the *Schrödinger-Weil* representation $\pi^m_{SW}$ with character $\psi^m(x) = \mathrm{e}^{2\pi\mathrm{i}mx}$, $m \in \mathbb{R}$.
$\mathbf{K}_+ = \frac{1}{2}(\mathbf{a}^\dagger)^2$, $k = 1/4$,

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# Continuation

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# Giulio Chiribella et all

**Quantum Benchmarks for Pure Single-Mode Gaussian States**

PRL January 2014

Keywords: quantum teleportation, quantum state storage, realized on different physical systems.

Aim: "realization of a perfect identity channel between an unknown input state $\psi_{in}$ at Alice and output state received by Bob.
Input-output fidelity $\mathcal{F} =_{in}< \psi|\rho|\psi >_{in}$, averaged over the ensemble of possible input states. To assess whether the execution of transmission protocols takes advantage of genuine quantum resources, it is mandatory to establish benchmarks for the average fidelity.
What is the general benchmark for teleportation and storage of arbitrary pure single - mode Gaussian states? We solve the longstanding problem of defining quantum benchmarks for general pure Gaussian single-mode states with arbitrary phase, displacement, and squeezing, randomly sampled according to a realistic prior distribution"
Pure single-mode gaussian states: $|\Psi >= D(\alpha)S(\xi)|0>$

# Mathieu Molitor

**Gaussian distributions, Jacobi group and Siegel-Jacobi space**, JMP Dec 2014

"Physics": study the measure in which the quantum formalism may have an information-theoretical origin.This idea is suggested by the observation that the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ is recovered from a purely statistical object, studied by Molitor in a previous publication. This bold approach to quantum mechanics via information geometry goes further than the geometric formulation of quantum mechanics due to Kibble, Cirelli et all, Heslot ..
The main contribution: link between the standard Gaussian distribution and the Siegel-Jacobi space.
The family $\mathcal{N}$ of Gaussian distributions functions $p(x;\mu,\sigma) := \frac{1}{\sqrt{2\pi\sigma}}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$, $x,\mu,\ \sigma \in \mathbb{R}, \mu > 0$, as a 2-dimensional *statistical manifold*. The tangent bundle $T\mathcal{N}$ has a natural (complex) 2-dimensional Kähler structure.

The manifold $T\mathcal{N} \equiv \mathcal{X}_1^J$.
Dombrowski's construction(*The geometry of tangent bundle*, 1962)): a dualistic structure $(h, \nabla, \nabla^*)$ is introduced on the manifold $M$ with affine connection $\nabla$. On $TM$: almost Hermitian structure $(g, J, \omega)$, associated to $(h, \nabla)$- $h$ - Riemannian metric on $M$, $g$ - Kähler metric, $\omega$ - fundamental two-form, $J$ - the almost complex structure, and the Ricci and the scalar curvature.
The space $\mathcal{K}(N)$ of Kähler functions (Cirelli) $f : N \rightarrow \mathbb{R} =$ functions whose Hamiltonian vector field $X_f = i_\omega df$ are Killing vectors on Kähler manifolds $N$ $(g, J, \omega)$, i.e. $L_{X_f} g = 0$.
Information Geometry (*Methods of information geometry*, Amari and Nagaoka 1993/2000): statistical manifold, Fisher metric and $\nabla^\alpha$-connections. The set $\mathcal{N}$ of Gaussian distributions $p(x; \mu, \sigma)$- as statistical manifold - parametrized as exponential family with two parameters $\theta = (\theta_1, \theta_2) = (\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2})$.

$\mathcal{X}_1^J$ is endowed with a particular homogeneous metric, $g_{KB}$, (Erich Kähler & Rolf Berndt) in $(u, v, x, y) \in \mathbb{R}^4$, $(\tau, z) \in \mathcal{X}_1 \times \mathbb{C}$, $\tau = u + iv; z = x + iy$. Expression in the coordinates $\theta, \dot{\theta}$ for $T\mathcal{N}$: the Fisher metric, Christoffel symbols and $g, J, \omega$. Result: $T\mathcal{N} \simeq \mathcal{X}_1^J$. The group of holomorphic isometries of $T\mathcal{N}$ - the affine symplectic group $G^{AS}$- proof based on 2-dimensional Eikonal equation. Description of the group of isometries of $T\mathcal{N}$ as $\mathrm{SL}^{\pm}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ . $\xi_M$ - the fundamental vector field associated with $\xi \in \mathfrak{g}$ of the Lie group $G$ acting on the manifold $M$; $\mathfrak{as}$ - the Lie algebra of $G^{AS}$, $\iota(M)$ the space of Killing vector fields of a Riemannian manifold $M$. Then the map $\mathfrak{as} \to \iota(T\mathcal{N})$, $\xi \to \xi_{T\mathcal{N}}$ is an anti-isomorphism of Lie algebras.

*Kähler functions and momentum map*: a linear map $\psi : \mathfrak{g}_1^J \to \mathbb{C}^{\infty}(\mathcal{X}_1^J)$ is explicitly defined. It is proved that: the map $\mathbf{J} : \mathcal{X}_1^J \to (\mathfrak{g}^J)^*$, $\mathbf{J}(p)(L) := \psi(L)(p)$, $p \in \mathcal{X}^J$, $L \in \mathfrak{g}_1^J$ is an equivariant momentum map; the map $\psi : \mathfrak{g}_1^J \to \mathcal{K}(\mathcal{X}_1^J)$ is a Lie algebra isomorphism; as symplectic manifold, $\mathcal{X}_1^J$ is a coadjoint orbit of the Jacobi group $G_1^J$.

# Benjamin Cahen

*Global Parametrization of Scalar Holomorphic Coadjoint Orbits of a Quasi-Hermitian Lie Group*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, (2013);
*Stratonovich-Weyl correspondence for the Jacobi group*, Communications in Mathematics, The University of Ostrava, (2014)

Different aspects of Berezin's quantization- Berezin symbols of $\pi(g)$, $g \in G$ and $d\pi(X)$, $X \in \mathfrak{g}$ -$\pi$ - a unitary irreducible representation of $G$ holomorphically induced from a character of $H$.
*Berezin quantization on generalized flag manifolds, Berezin quantization for discrete series, Stratonovich-Weyl correspondence for discrete series representations*. In *Berezin quantization and holomorphic representations* : $G =$ quasi-Hermitian Lie group, $\pi$ a unitary highest weight representation realized in a reproducing Hilbert space of holomorphic functions (Neeb).

Quasi-Hermitian groups- include compact, connected, simply-connected Lie groups, with the their duals - the noncompact Hermitian groups, with associated homogeneous manifold the noncompact hermitian symmetric spaces $\mathcal{D} = G/H$; semidirect products of semisimple Lie groups with nilpotent Lie groups, the Jacobi group.
A diffeomorphism $\psi$ from $\mathcal{D}$ to the coadjoint orbit $\mathcal{O}(\xi_0)$ is established.
Examples: $\mathrm{SU}(p, q)$ , $G^J_n$. The Stratonovich-Weyl correspondence for $G^J_1$.
The Berezin transform is extended such that can be applied to the derived representation $d\pi_{\chi}(X)$ for $X \in \mathfrak{g}^J_1$. This permits to obtain the Stratonovich-Weyl symbols for the derived representation of the Jacobi group $G^J_1$.

# Sepanski M. R at all

Sepanski, M. R.; Stanke, R.J. *Global Lie symmetries of the heat and Schrödinger equation*, J. Lie Theory 2010
Certain spaces of global solutions for the family of differential equations $4s\partial_t + \Delta_n = 0$, invariant under the group $(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathrm{O}(n)) \ltimes H_{2n+1})$.
$s = -1/4$ -the heat eq. $s = i/4$ - Schr. eq.

Franco, Jose, *Global* $\widetilde{\mathrm{SL}(2,R)}$ *representations of the Schrödinger equation with singular potential* , Centr. Eur. J. Math. 2012
Certain spaces of global solutions of the Schrödinger equation with singular potential, $2iu_t + u_{xx} = \frac{2\lambda}{x^2}u$. The solution of the Schrödinger equation is constructed via the theory of induced representation of a certain ("parabolic") subgroup $\bar{P}$ of the covering $G$ of the Jacobi group, $\widetilde{\mathrm{SL}(2,\mathbb{R})} \ltimes H_3$.The two-fold cover of the $\mathrm{SL}(2,\mathbb{R})$ is realized as in M. Kashiwara; M. Vergne, Invent. Math. 44 (1978). Follows Sepanski and Stanke (2010)

Franco, Jose A.; Sepanski, Mark R, *Global representations of the heat and Schrödinger equation with singular potential*, Electronic J. Differential equations, (2013)

Special subspaces of solutions of the family of differential equations $4s\partial_t + \Delta_n - \frac{2\lambda}{||x||^2} = 0$ which are invariant under the action of the group $G_1 = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathrm{O}(n)$ or $G = G_1 \ltimes H_{2n+1}$.

Jose A. Franco, *Globalization of the Actions of the Lie Symmetries of Nonlinear Wave Equations with Dissipation* to appear in European Journal of Mathematics, 2015

Global solutions of a family of nonlinear wave equation with dissipations in 1+1 dimensions of the form $u_{tt} + \varphi(u)u_t = (f(u)u_x)_x$.

**In all these papers**: The general symmetry method for analyzing solutions of differential equations using Lie's prolongation method extended to find global solutions by restricting to a special subset of the solution space. The group of symmetries $G$ of the differential equation is determined, then a parabolic subgroup $P$ is fixed, and the solution is constructed via the induced representations of $G$ from a character of $P$.

**M. HUNZIKER, M. R. SEPANSKI, R. J. STANKE**, *GLOBAL LIE SYMMETRIES OF A SYSTEM OF SCHRÖD. EQUATIONS AND THE OSCILLATOR REPRESENTATION*, Miskolc Mathematical Notes HU (2013)

$$4s\partial_{t_{ij}}f(x,t)+\partial^2_{x_i}f(x,t)=0, 1\leq i\leq n$$
$$2s\partial_{t_{ij}}f(x,t)+\partial_{x_i}\partial_{x_j}f(x,t)=0, 1\leq i\leq j\leq n,$$

$s\in i\mathbb{R}^x$, $x=(x_i)\in\mathbb{R}^n$, $t$- real symmetric matrix. Lie prolongations method calculates the infinitesimal symmetries as $\mathfrak{g}^J_n$. The space of functions satisfying eqs. carries a representation of $\mathfrak{g}^J_n$. On some subspaces of $\mathcal{C}^\infty(\mathbb{R}^n\times \mathrm{Sym}(n,\mathbb{R}))$ it is obtained the exponentiation to the covering of the Jacobi group $G^J_n$.- the oscillator representation of Kashiwara Vergne (metaplectic, Segal-Shale-Weil representation). In Physics: squeezed states.

Berceanu & Gheorghe: $\exp(\alpha z+\frac{1}{2}wz^2)=\sum_{n\geq 0}\frac{z^n}{\sqrt{n!}}f_n(\alpha,w)$ Any solution $f\in\mathcal{H}_{\rm hol}(\mathbb{C}\times\mathcal{D})$ of $(\partial^2/\partial\alpha^2-2\partial/\partial w)f=0$ can be written as $f=\sum_{n\geq 0}c_nf_n$. The reproducing kernel can be written in a series of polynomials obtained from $f_n$.

Giuseppe Marmo, Peter W. Michor, Yury A. Neretin *The Lagrangian Radon Transform and the Weil Representation* J Fourier Anal Appl (2014) Quantum tomography, lagrangian Radon Transform, Weil representation, affine symplectic group.

# Balanced metric - comments

$$f(z) = \ln K_M(z) = \ln(e_{\bar{z}}, e_{\bar{z}})$$

• Donaldson; Arezzo & Loi,
• Berezin quantization of Kähler manifolds- in fact: $\mathbb{C}^n$, classical bounded domains (=Hermitian symmetric spaces); Loi & Mossa: Berezin quantization on homogeneous bounded domains
• $\epsilon$-function: Rawnsley, Cahen, Gutt; globalization of Berezin's construction to non-homogeneous Kähler manifolds; Berezin quantization via coherent states, a particular realization of geometric quantization.
• Geometric aspect of Berezin's quantization of homogeneous **manifolds** is different of Bergman construction on **domains**.

# Balanced Kähler two-form on Siegel-Jacobi ball

The homogeneous Kähler two-form $\omega$

$$-\mathrm{i}\omega(z,W)_{k\mu} = \frac{k}{2}\mathrm{Tr}(B\wedge\bar{B}) + \mu\mathrm{Tr}(A^t\bar{M}\wedge\bar{A}), A = \mathrm{d}\, z + \mathrm{d}\, W\bar{\eta},$$
$$B = M\,\mathrm{d}\, W, M = (\mathbb{1}_n - W\bar{W})^{-1},\ \eta = M(z + W\bar{z}).$$

Comment:S. B. 2007, 2008; R. Berndt, 1982, E. Kähler 1983, 1992 ( $\mathcal{X}_n$, $n=1$); J-H. Yang 2010 ... $(n,m)$

Remark: $FC(\eta,W) = (z,W) \in \mathbb{C}^n \times \mathrm{Sp(n,\mathbb{R})_{\mathbb{C}}}$ ; S.B. 2012, 2014

# The fundamental conjecture (Gindikin-Vinberg)

*Every homogenous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogenous manifold (generalized flag manifold), a homogenous bounded domain and* $\mathbb{C}^n/\Gamma$, *where* $\Gamma$ *denotes a discrete subgroup of translations of* $\mathbb{C}^n$.

$$\begin{array}{ccccc} M = (G^{\mathbb{C}}/P) & \times & D & \times & (\mathbb{C}^n/\Gamma) \\ \swarrow & & \downarrow & & \downarrow \\ \text{flag manifold} & & \text{homogeneous} & & \text{Kähler} \\ P - \text{parabolic} & & \text{bounded domain} & & \text{flat} \end{array}$$

## Proposition

*The FC-transform,* $FC(\eta, W) = (z, W)$, $z = \eta - W\bar{\eta}$, *is a homogeneous Kähler diffeomorphism,* $FC^*\omega_{k\mu}(z, W) = \omega_{k\mu}(\eta, W)$:

$$\omega_{k\mu}(\eta, W) = \omega_k(W) + \omega_\mu(\eta).$$

$\omega_{k\mu}(\eta, W)$ *is invariant to the action of* $G^J_n$ *on* $\mathbb{C}^n \times \mathcal{D}_n$,
$((g, \alpha), (\eta, W)) \rightarrow (\eta_1, W_1)$,

$$\eta_1 = p(\eta + \alpha) + q(\bar{\eta} + \bar{\alpha}),$$

Comment:S. B. 2012, Rev. Math. P

# Geometry of $\mathcal{D}_n^J$

•Jacobi groups - unimodular, **non**reductive, algebraic gr. of Harish-Chandra type (Satake);

•The Siegel-Jacobi spaces - reductive, **non**symmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding. **Not** Einstein manifold w.r. balanced metric $\omega_{k\mu}(z,W)$, but Einstein w.r. Bergman metric, $f=\ln\mathcal{G}$, $\mathcal{G}=\det h$. $\Delta_{\mathcal{D}_1^J}(\ln(\mathcal{G}(z,w)))=-s(z,w)=\frac{3}{2k}$

•With the orthonormal basis in which the reproducing kernel $K$ can be developed an effective (Kobayashi) embedding of Siegel-Jacobi manifold in $\mathbb{CP}^\infty$ can be constructed. $\iota:\mathfrak{D}_n^J\hookrightarrow\mathbb{CP}^\infty$ , $\iota=[\varphi_0:\varphi_1:\dots]$ $\omega_n|_{\mathfrak{D}_n^J}=\iota^*\omega_{FS}|_{\mathbb{CP}^\infty}$. The canonical projection $\xi:\mathfrak{H}\setminus 0\rightarrow\mathbb{P}(\mathfrak{H})$ is $\xi(\mathbf{z})=[\mathbf{z}]$ and the hermitian metric on $\mathbb{CP}^\infty$ is the Fubini-Studi metric $\mathrm{d}\,s^2|_{FS}([z])=\frac{(\mathrm{d}\,\mathbf{Z},\mathrm{d}\,\mathbf{Z})(\mathbf{Z},\mathbf{Z})-(\mathrm{d}\,\mathbf{Z},\mathbf{Z})(\mathbf{Z},\mathrm{d}\,\mathbf{Z})}{(\mathbf{Z},\mathbf{Z})^2}$. *The Siegel-Jacobi disk $\mathfrak{D}_n^J$ is a coherent state manifold*, i.e. it is a submanifold of a (infinite-dimensional) projective Hilbert space, and *the Jacobi group $G_n^J$ is a coherent-state group* in the meaning of Lisiecki and Neeb.

Comment: S. B. 2014 IJGMMP + arXIve 2014.

# Dequantization

The time-dependent Schrödinger equation on $\mathfrak{H}$:

$$\mathbf{H}(t)\psi(t) = \mathrm{i}\hbar\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t},$$

Passing on from the dynamical system problem in the Hilbert space $\mathfrak{H}$ to the corresponding one on $M = G/H$ is called *dequantization*, and the system on $M$ is a classical one. The motion on the classical phase space can be described by the local equations of motion:

$$-\mathrm{i}\dot{z}_\alpha = \{\mathcal{H}, z_\alpha\}, \mathcal{H} = \frac{(e_z, \mathbf{H}e_z)}{(e_z, e_z)}, e_z = \exp(\sum_{\alpha\in\Delta_+} z_\alpha \mathbf{X}_\alpha) \in \mathfrak{H}, z_\alpha \in M,$$

$$\omega = \mathrm{i}\sum_{\alpha\in\Delta_+} g_{\alpha\beta}\,\mathrm{d}\,z_\alpha \wedge \mathrm{d}\,\bar{z}_\beta; g_{\alpha\beta} = \frac{\partial^2}{\partial z_\alpha \partial z_{\bar{\beta}}} f,\ \ f = \ln(e_z, e_z).$$

Solution of the Schrödinger equation of motion $\psi(t) = \mathrm{e}^{\mathrm{i}\phi} \frac{e_z}{\sqrt{(e_z, e_z)}}$.

## Proposition

*Suppose:* $\mathbf{H} = \sum_{\lambda\in\Delta} \epsilon_\lambda \mathbf{X}_\lambda$; $\mathbb{X}_\lambda = P_\lambda + \sum_\beta Q_{\lambda,\beta}\partial_\beta . \Longrightarrow$
*The classical & quantum equations of motion on* $M = G/H$

$$\begin{aligned} \mathrm{i}\dot{z}_\alpha &= \sum_\lambda \epsilon_\lambda Q_{\lambda,\alpha}, \\ \phi &= \phi_D + \phi_B; \\ \dot{\phi}_D &= -\mathcal{H}(t); \\ \dot{\phi}_B &= \frac{\mathrm{i}}{2} \sum_{\alpha\in\Delta_+} (\dot{z}_\alpha \partial_\alpha - \dot{\bar{z}}_\alpha \bar{\partial}_\alpha) \ln(e_z, e_z). \end{aligned}$$

Commet:S. B.& A. Gheorghe, J. M. P 1992; S. B & L. Boutet de Monvel, 1993; S. B. 2012 Rev. Math. P

# Realization of $\mathfrak{g}^J_n$ in holomorphic first order diff. operators with polynomial coefficients on $\mathcal{D}^J_n$

## Lemma

$$\mathbf{a}_k = \frac{\partial}{\partial z_k}, \quad \mathbf{a}^{\dagger}_k = z_k + w_{ki}\frac{\partial}{\partial z_i};$$

$$\mathbf{K}^0_{kl} = \frac{k_k}{4}\delta_{kl} + \frac{z_k}{2}\frac{\partial}{\partial z_l} + w_{ki}\nabla_{il}, \ \mathbf{K}^-_{kl} = \nabla_{kl};$$

$$\mathbf{K}^+_{kl} = \frac{k_k + k_l}{4}w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2}(z_l w_{ik} + z_k w_{il})\frac{\partial}{\partial z_i} + w_{\alpha l}w_{ki}\nabla_{i\alpha}.$$

$(\nabla_{ij})_{i,j=1,\dots,n} = (\chi_{ij}\frac{\partial}{\partial w_{ij}})_{i,j=1,\dots,n}, \ \chi_{ij} = \frac{1+\delta_{ij}}{2}$

# Hamiltonian- linear, hermitian

$$\mathbf{H} = \epsilon_i \mathbf{a}_i + \bar{\epsilon}_i \mathbf{a}_i^\dagger + \epsilon_{ij}^0 \mathbf{K}_{ij}^0 + \epsilon_{ij}^- \mathbf{K}_{ij}^- + \epsilon_{ij}^+ \mathbf{K}_{ij}^+ .$$

$$\epsilon_0^\dagger = \epsilon_0;\ \epsilon_- = \epsilon_-^t;\ \epsilon_+ = \epsilon_+^t;\ \epsilon_+^\dagger = \epsilon_- .$$

$$\epsilon_- = m + \mathrm{i}n,\ \epsilon_0^t/2 = p + \mathrm{i}q; p^t = p; m^t = m; n^t = n; q^t = -q.$$

# Equations on manifolds

Matrix Riccati equation on manifold

$$\dot{W} = AW + WD + B + WCW, \; A, B, C, D \in M(n, \mathbb{C}); \tag{4.3}$$

**Linearization**: $W = XY^{-1} \Leftrightarrow$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h \begin{pmatrix} X \\ Y \end{pmatrix}, \; h = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}. \tag{4.4}$$

*Every solution of* (4.4) *is a solution of* (4.3), *whenever* $\det(Y) \neq 0$. Linear differential eq for $z \in \mathbb{C}^n$

$$\dot{z} = M + Nz; \; M = E + WF; \; N = A + WC, \; E, F \in C^n.$$

# Classical motion & quantum evolution on the Siegel-Jacobi ball $\mathcal{D}_n^J$

a) $(z, W) \in \mathbb{C}^n \times \mathcal{D}_n$

$$A_c = -\frac{\mathrm{i}}{2}\epsilon_0^t, \ B_c = -\mathrm{i}\epsilon_-, \ C_c = -\mathrm{i}\epsilon_+, \ D_c = A_c^t;$$
$$E_c = -\mathrm{i}\epsilon, \ F_c = -\mathrm{i}\bar{\epsilon}$$

$$\mathrm{i}\dot{W} = \epsilon_- + (W\epsilon_0)^s + W\epsilon_+W, \ W \in \mathcal{D}_n, \quad (4.6a)$$
$$\mathrm{i}\dot{z} = \epsilon + W\bar{\epsilon} + \frac{1}{2}\epsilon_0^t z + W\epsilon_+ z, \ z \in \mathbb{C}^n, \quad (4.6b)$$

b) The *FC* transform: $z = \eta - W\bar{\eta}$

$$\mathrm{i}\dot{\eta} = \epsilon + \epsilon_-\bar{\eta} + \frac{1}{2}\epsilon_0^t\eta, \ \eta \in \mathbb{C}^n. \quad (4.7)$$

# Continuation

c) The linear system of differential equations (4.4) attached to the matrix Riccati equation (4.6a) is

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_c \begin{pmatrix} X \\ Y \end{pmatrix}, \; h_c = \begin{pmatrix} -\mathrm{i}(\frac{\epsilon_0}{2})^t & -\mathrm{i}\epsilon_- \\ \mathrm{i}\epsilon_+ & \mathrm{i}\frac{\epsilon_0}{2} \end{pmatrix} \in \mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}}, \; X/Y \in \mathcal{D}_n.$$

d) In (4.7) introduce $\eta = \xi - \mathrm{i}\zeta$, $\xi, \zeta \in \mathbb{R}^n$, $\epsilon = b + \mathrm{i}a$, where $a, b \in \mathbb{R}^n$- a system of first order real differential equations with real coefficients,

$$\dot{Z} = h_r Z + F, \; Z = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \; F = \begin{pmatrix} a \\ b \end{pmatrix}, h_r = \begin{pmatrix} n+q & m-p \\ m+p & -n+q \end{pmatrix} \in \mathfrak{sp}(n,\mathbb{R})$$

Comment: S. B. 2012, Rev. Math. P

# The problem

Representing the solution of the equation

$$\frac{\mathrm{d}\, U(t)}{\mathrm{d}\, t} = A(t)U(t), \quad U(0) = I,$$

in the form of product of exponentials

$$U(t) = \prod_{i=1}^{n} \exp(\xi_i(t)X_i), \tag{4.8}$$

where $A$ and $U$ are linear operators,

$$A(t) = \sum_{i=1}^{n} \epsilon_i(t)X_i, \tag{4.9}$$

$\epsilon_i(t)$ are scalar functions of $t$ and $\{X_i\}_{i=1,\dots,n}$ are the generators of a Lie algebra $\mathfrak{g}$.

# Lemma

If $\{\mathbf{X}_1,\ldots,\mathbf{X}_n\}$ is a basis of a Lie algebra $\mathfrak{g}$, then

$$\left(\prod_{j=1}^{r}\exp(\xi_j\mathbf{X}_j)\right)\mathbf{X}_i\left(\prod_{j=r}^{1}\exp(-\xi_j\mathbf{X}_j)\right)=\sum_{k=1}^{n}\eta_{ki}\mathbf{X}_k,\;\; i,r=1,2,\ldots,n,$$

where $\eta_{ki}=\eta_{ki}(\xi_1,\ldots,\xi_r)$ is an analytic function of $\xi_1,\ldots,\xi_r$. If $A(t)$ is given by (4.9), then there exists a neighborhood of $t=0$ in which the solution of the equation

$$\dot{U}(t)=A(t)U(t),\quad U(0)=I$$

may be expressed in the form (4.8), where the $\xi_i(t)$ are scalar functions of $t$. Moreover, $\xi^t:=(\xi_1,\ldots\xi_n)$ satisfy the first order differential equation

$$\eta\dot{\xi}=\epsilon,$$

which depend only on the Lie algebra $\mathfrak{g}$ and the $\epsilon(t)$'s.

# Wei-Norman & Berezin's equations of motion

## Proposition

*The quantum and classical Berezin's equations of motion on the Siegel-Jacobi disk determined by a hermitian Hamiltonian, linear in the generators of Jacobi group $G_1^J$, expressed in the FC-coordinates, are the same as the equations obtained applying the Wei-Norman method.*

Comment: A. Gheorghe, *Quantum and classical Lie systems for extended symplectic groups*, Romanian J. Phys. **58** (2013) 1436–1445.
S. B 2014, arXive; proof: Lemma Wei-Norman;

$$U^{-1}(\xi)\dot{U}(\xi) = \sum_{i=1}^{n} \dot{\xi}_i \mathbf{Y}_i, \quad \mathbf{Y}_i = \left(\prod_{k=n}^{i+1}[\exp(-\xi_k \mathrm{ad}\mathbf{X}_k)]\right)\mathbf{X}_i$$